

Calculus

CHAPTER 9

Area

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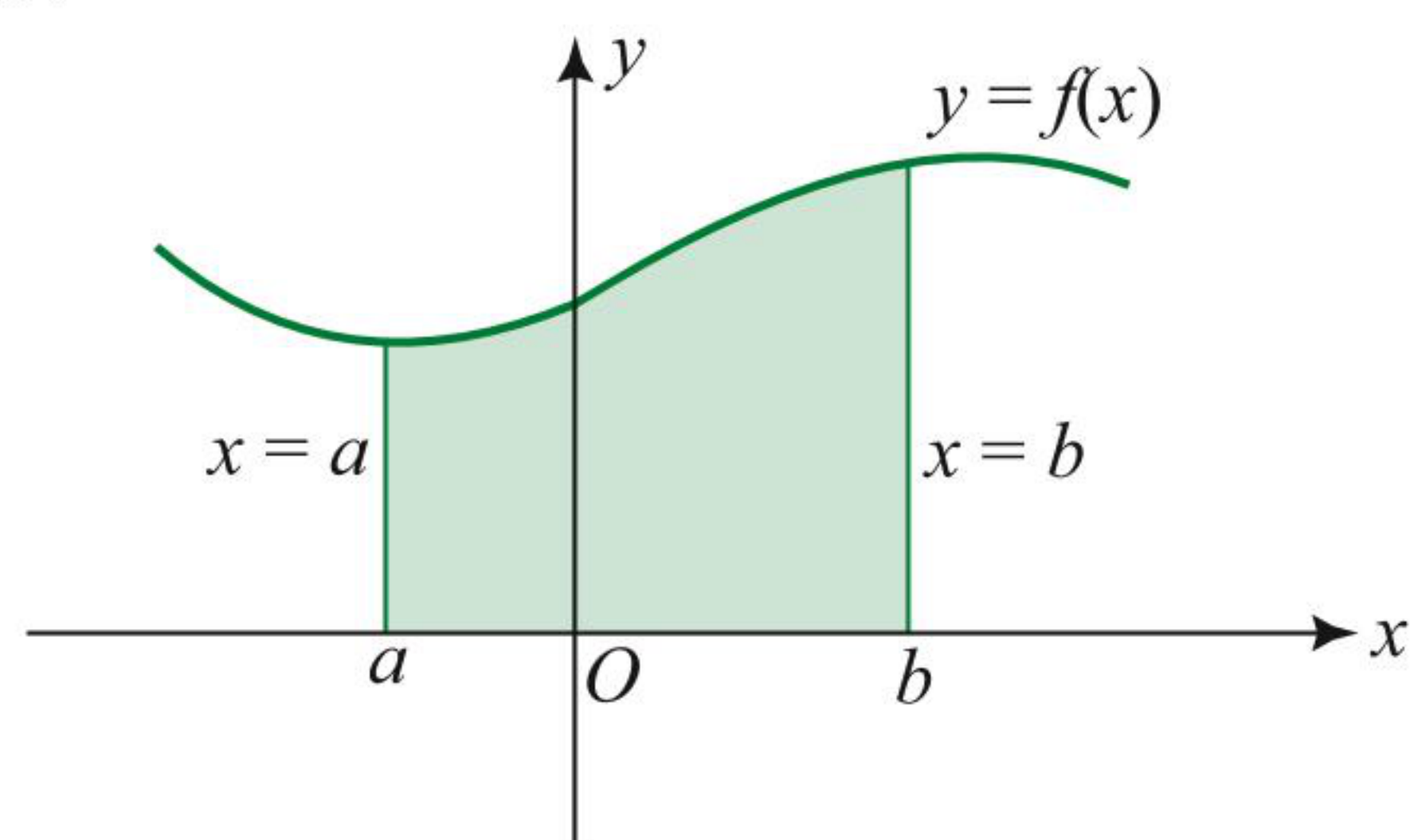
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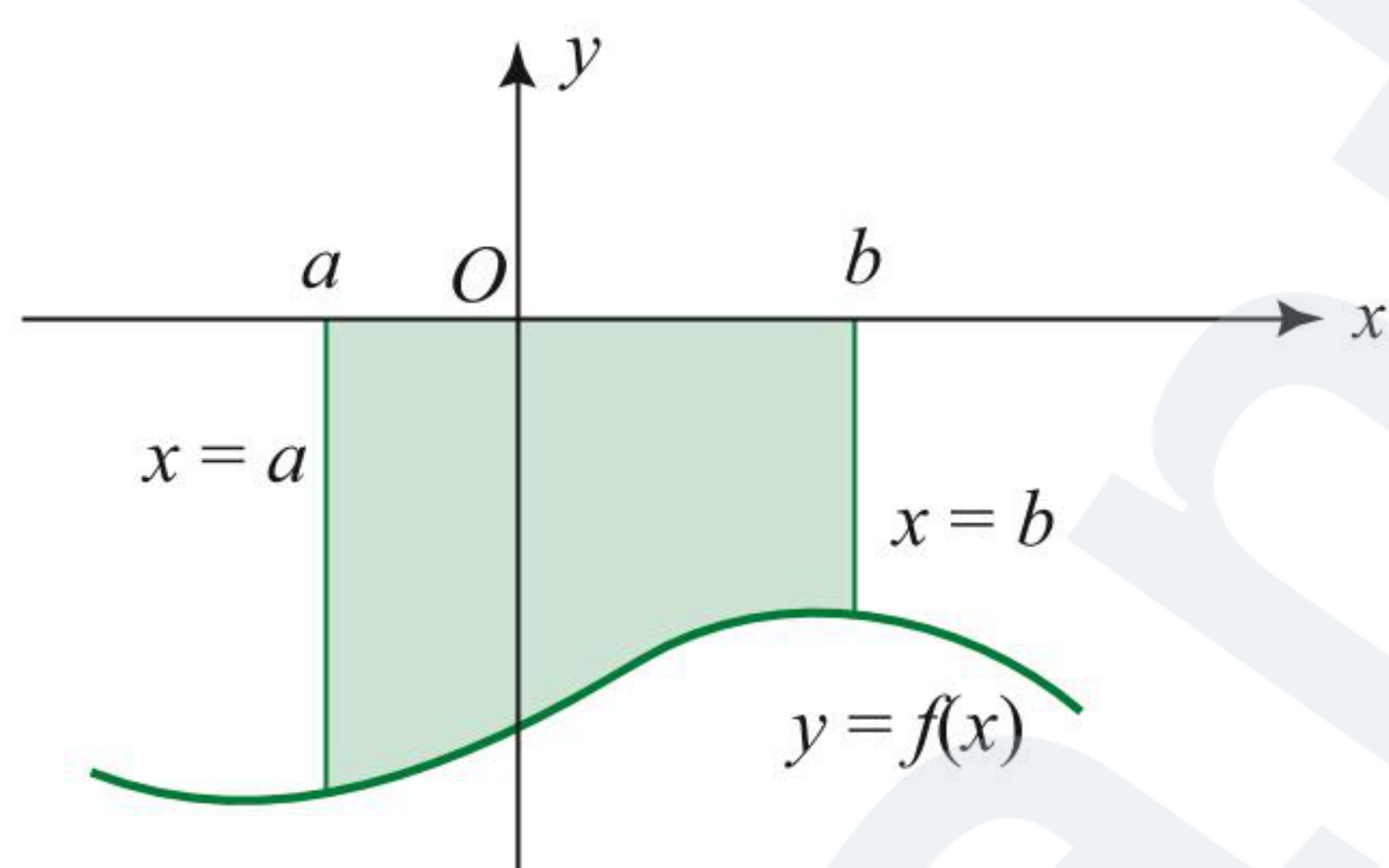
Area

AREA BOUNDED BY CURVE AND AXIS

In definite integration we learned that if $f(x) \geq 0 \forall x \in [a, b]$ then $\int_a^b f(x)dx$ represents the area bounded by $y=f(x)$, x -axis and lines $x=a$ and $x=b$.



If $f(x) \leq 0 \forall x \in [a, b]$ then area bounded by $y=f(x)$, x -axis and lines $x=a$ and $x=b$ is $\left| \int_a^b f(x)dx \right|$



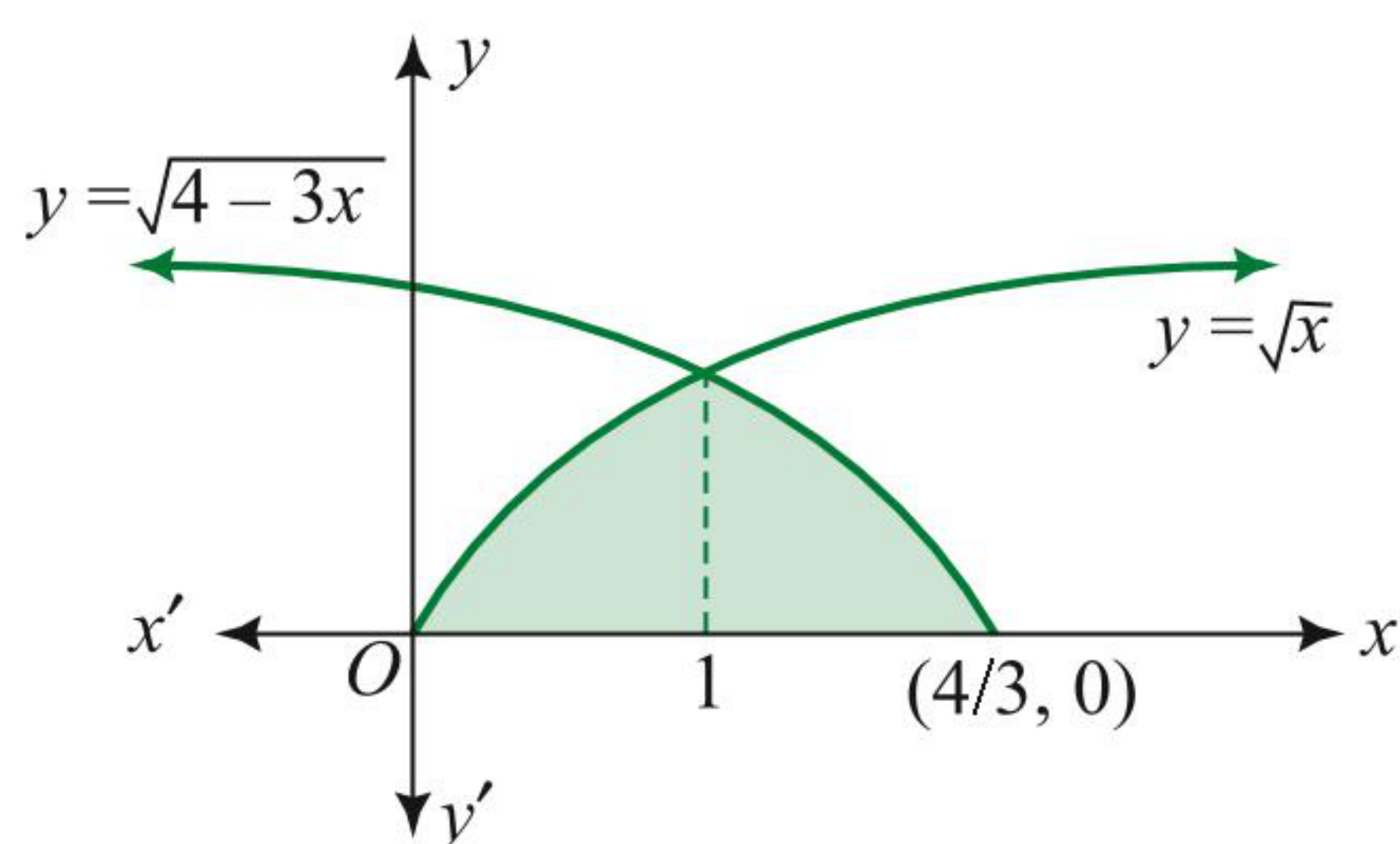
Note:

- Area bounded by $y = \sin x$ and x -axis between $x=0$ and $x=\pi$ is 2 sq. units.
- Area bounded by $x=0$, $y=0$ and $y = \log_e x$ is 1 sq. units.

ILLUSTRATION 9.1

Find the area of the closed figure bounded by the curves $y = \sqrt{x}$, $y = \sqrt{4-3x}$, and $y=0$.

Sol.

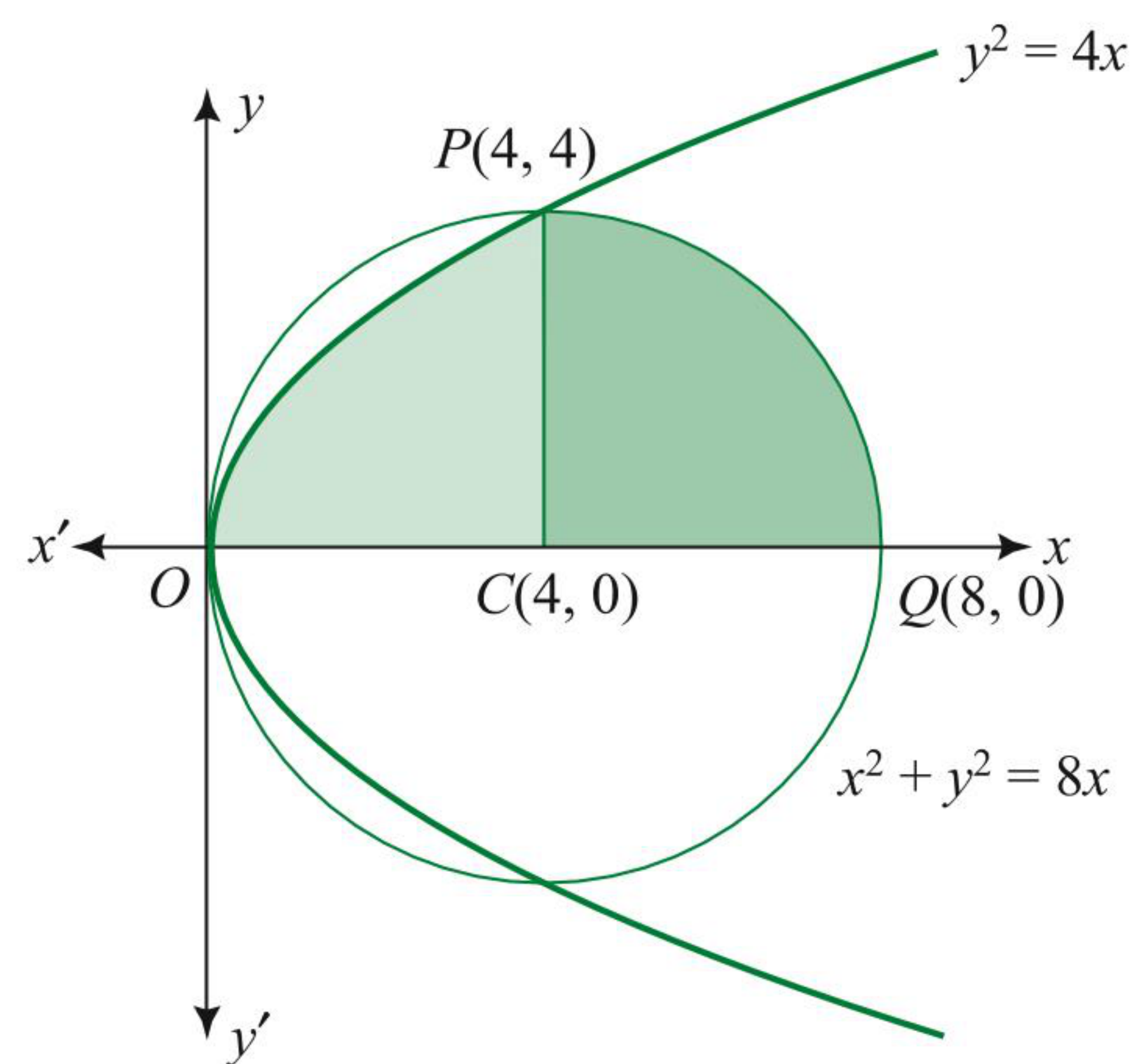


$$\begin{aligned} A &= \int_0^1 (\sqrt{x}dx) + \int_1^{4/3} \sqrt{4-3x}dx \\ &= \left(\frac{x^{3/2}}{3/2} \right)_0^1 + \left(\frac{(4-3x)^{3/2}}{-3(3/2)} \right)_1^{4/3} \\ &= \frac{2}{3} + \frac{2}{3} \left[\frac{1}{3} \right] \\ &= \frac{2}{3} + \frac{2}{9} = \frac{8}{9} \text{ sq. units} \end{aligned}$$

ILLUSTRATION 9.2

Find the area, lying above the x -axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Sol. Solving the curves, we get $x^2 + 4x = 8x \Rightarrow x = 0, 4$



$$\text{Required area} = \int_0^4 y_{\text{parabola}} dx + \int_4^8 y_{\text{circle}} dx$$

Circle is $(x-4)^2 + y^2 = 4^2$,

$$\text{Area of quarter of circle} = \frac{1}{4} \pi 4^2 = 4\pi$$

$$\begin{aligned} A &= \int_0^4 2\sqrt{x} dx + 4\pi \\ &= \frac{4}{3} \left[x^{3/2} \right]_0^4 + 4\pi \\ &= \frac{4}{3} \times 4\sqrt{4} + 4\pi \text{ sq. units} \\ &= \frac{32}{3} + 4\pi \text{ sq. units} \end{aligned}$$

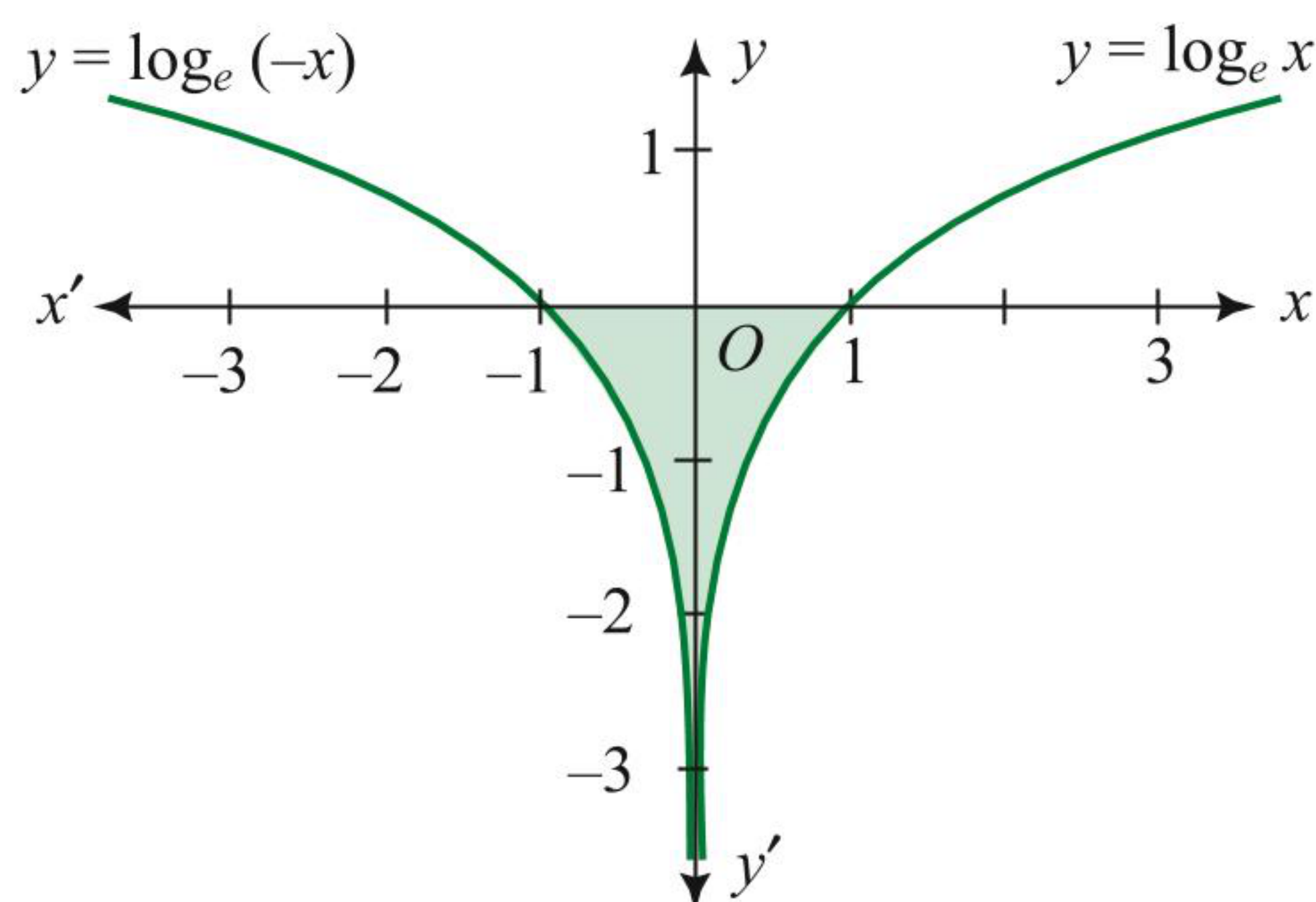
ILLUSTRATION 9.3

Find the area bounded by

- (i) $y = \log_e |x|$ and $y = 0$ (ii) $y = |\log_e |x||$ and $y = 0$

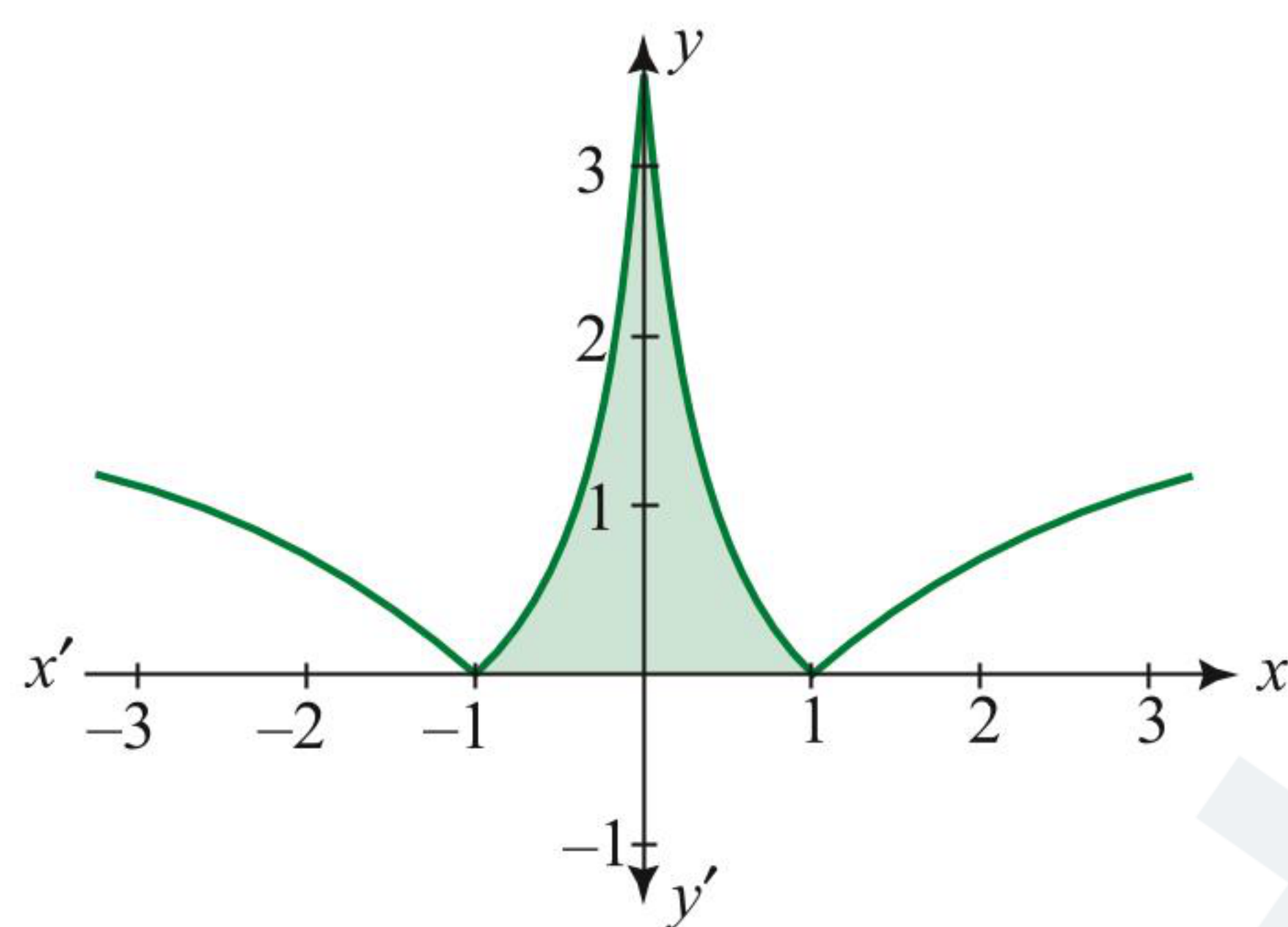
Sol.

- (i)
- $y = \log_e |x|$
- and
- $y = 0$



From the figure, required area = area of the shaded region
 $= 1 + 1 = 2$ sq. units (as we know that area bounded by
 $y = \log_e x$, $x = 0$, and $y = 0$ is 1 sq. units)

- (ii)
- $y = |\log_e |x||$
- and
- $y = 0$



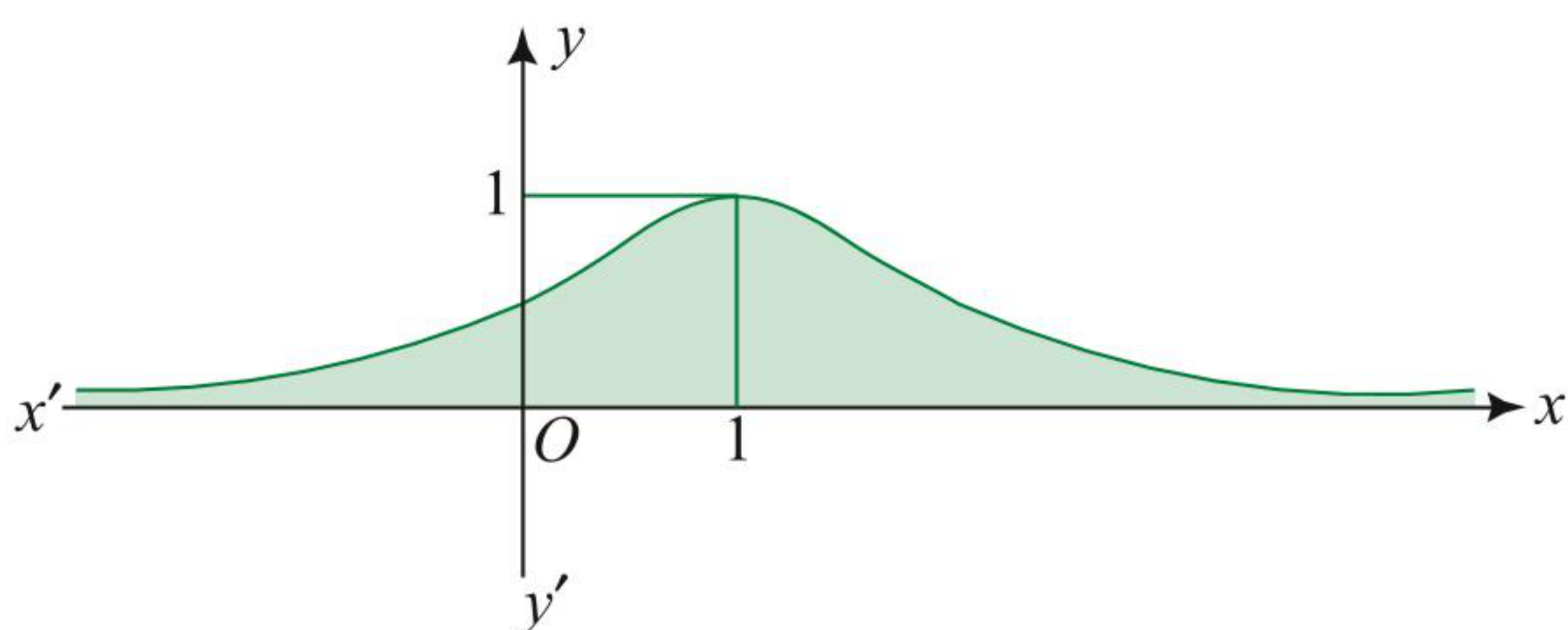
From the figure,
 Required area = Area of the shaded region
 $= 1 + 1 = 2$ sq. units.

ILLUSTRATION 9.4Find the area bounded by $y = \frac{1}{x^2 - 2x + 2}$ and x -axis.

Sol. $y = \frac{1}{(x-1)^2 + 1}$

When $x = 1$, $y_{\max.} = 1$ When $x \rightarrow \pm \infty$, $y \rightarrow 0$ Therefore, x -axis is the asymptote.Also $f(1+x) = f(1-x)$ Hence, the graph is symmetrical about line $x = 1$

From these information the graph of function is as shown in the figure.



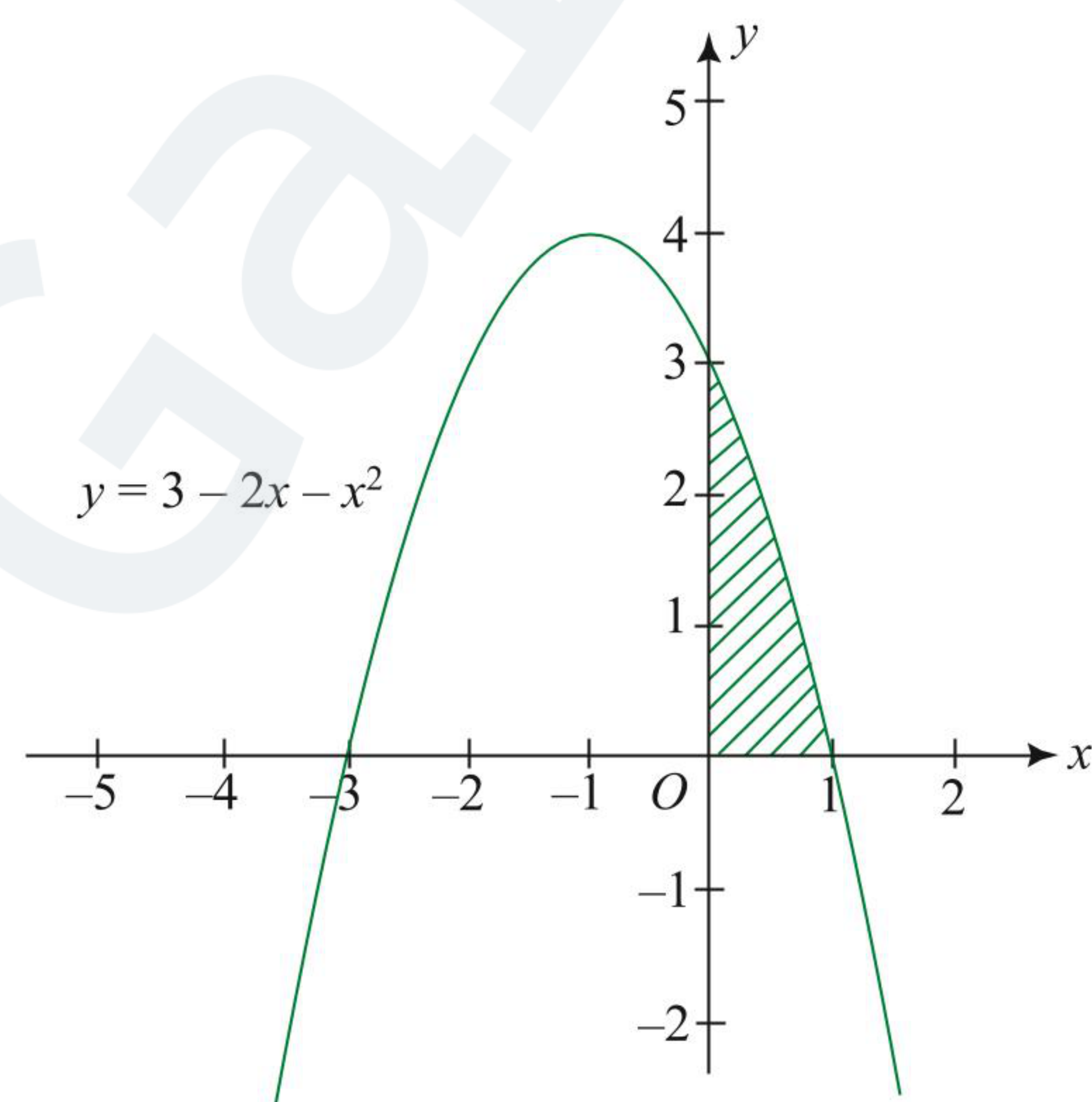
$$\text{Area} = 2 \int_1^{\infty} \frac{1}{(x-1)^2 + 1} dx = [2 \tan^{-1}(x-1)]_1^{\infty} = \pi \text{ sq. units}$$

ILLUSTRATION 9.5The area of the region for which $0 < y < 3 - 2x - x^2$ and $x > 0$ is

Sol. $0 < y < 3 - 2x - x^2$, $x > 0$

Consider parabola $y = 3 - 2x - x^2$

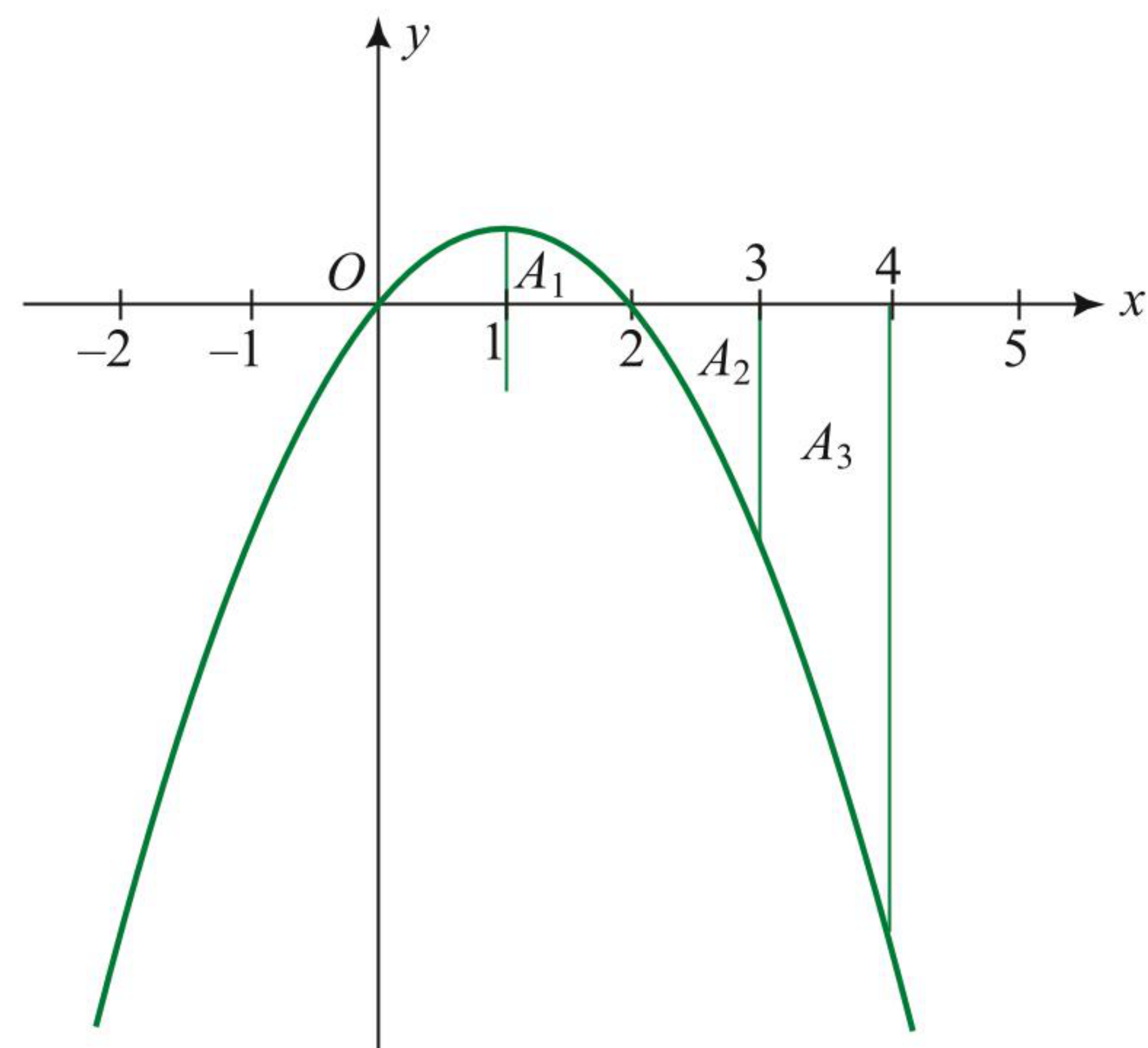
or $(x+1)^2 = -(y-2)$

The parabola having vertex $(-1, 2)$ and concave downward.For $x, y > 0$, the points lie inside parabola in the first quadrant as shown in the following figure.

$$\begin{aligned} \therefore \text{Required area} &= \int_0^1 (3 - 2x - x^2) dx \\ &= \left[3x - x^2 - \frac{x^3}{3} \right]_0^1 = \frac{5}{3} \text{ sq. units} \end{aligned}$$

ILLUSTRATION 9.6If the area bounded between x -axis and the curve of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 square units then prove that $a \in (3, 4)$ or $a \in (-2, -1)$.

Sol. We have $y = 3x(2-x)$



Graph of the function is downward parabola intersecting x -axis at $x = 0$ and $x = 2$.

$$\begin{aligned}\text{Now } I &= \int (6x - 3x^2) dx = \frac{6x^2}{2} - \frac{3x^3}{3} \\ &= 3x^2 - x^3 \\ &= x^2(3 - x) = F(x)\end{aligned}$$

$$\begin{aligned}\text{Now } A_1 &= F(2) - F(1) = 4 - 2 = 2 \text{ sq. units} \\ A_2 &= F(2) - F(3) = 4 - 0 = 4 \text{ sq. units} \\ A_3 &= F(3) - F(4) = 0 - (-16) = 16 \text{ sq. units}\end{aligned}$$

Since $A_1 + A_2 + A_3 = 22$ sq. units.

So, $a \in (3, 4)$

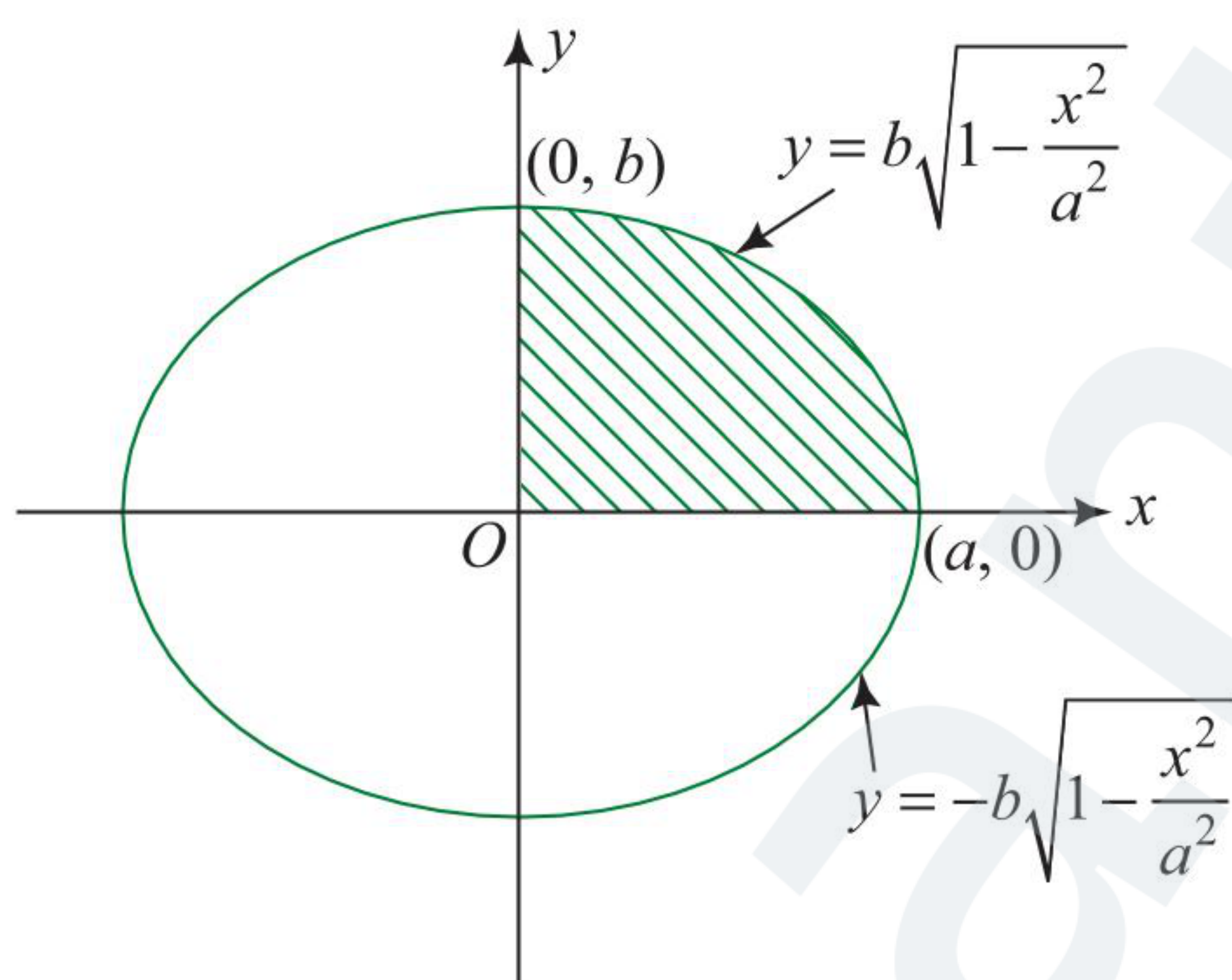
From the symmetric of the parabola, other value of a lies in $(-2, -1)$

ILLUSTRATION 9.7

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also prove that the area common to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ and its auxiliary circle $x^2 + y^2 = a^2$ is equal to the area of another ellipse whose semi axes are a and $a - b$.

Sol. We have ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\therefore y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$



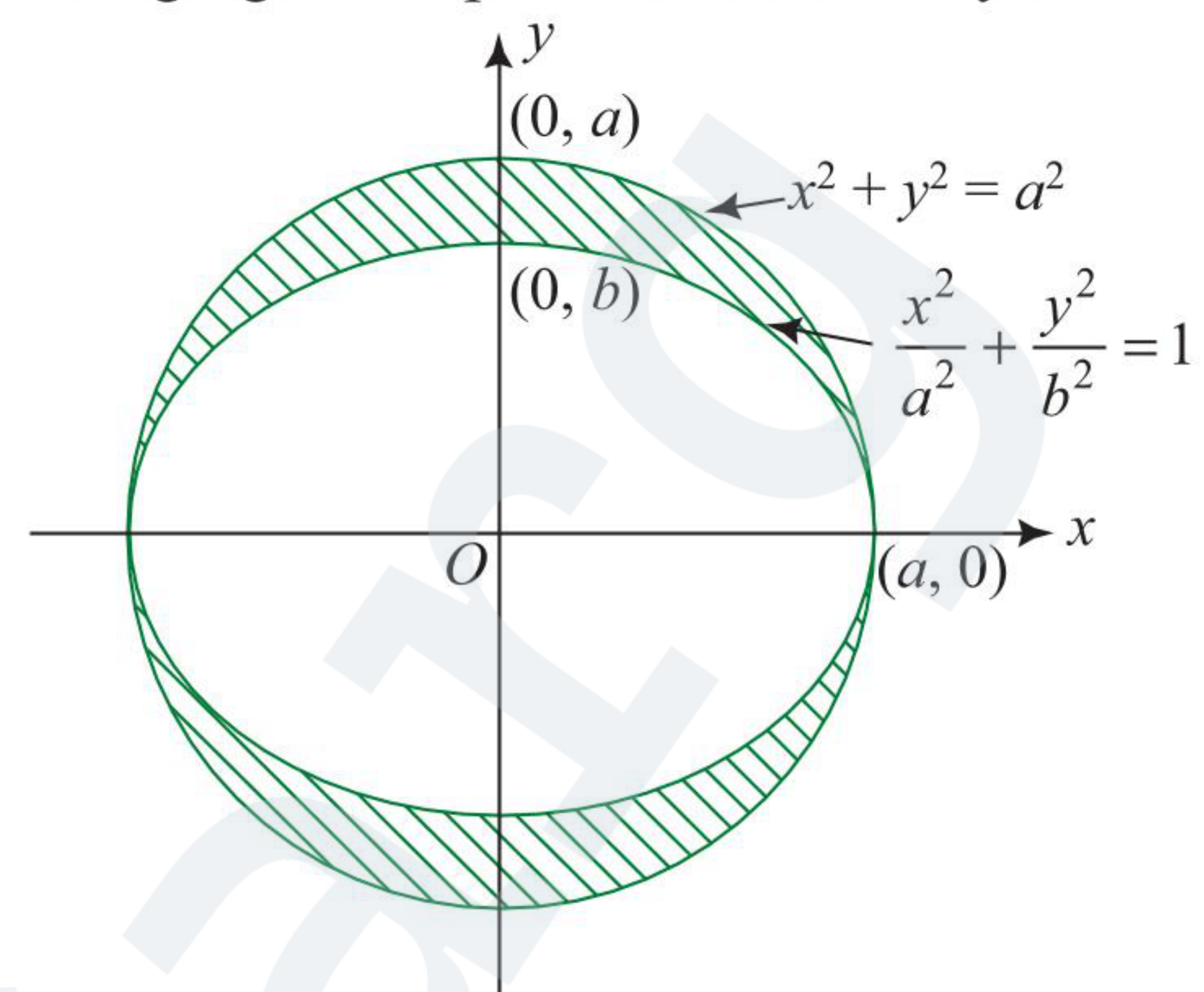
Since graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetrical about both the axis,

Area enclosed by ellipse = $4 \times$ Area in the first quadrant

$$\begin{aligned}&= 4 \times \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} dx \\ &= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\ &= \frac{4b}{a} \left[0 + \frac{a^2}{2} \sin^{-1} 1 - 0 - 0 \right]\end{aligned}$$

$$\begin{aligned}&= \frac{4b}{a} \left[\frac{a^2}{2} \cdot \frac{\pi}{2} \right] \\ &= \pi ab\end{aligned}$$

In the following figure ellipse and its auxiliary are drawn.



Area of the ellipse = πab

Area of the circle = πa^2

Common area between curves is that shown as shaded region

$$\begin{aligned}&= \text{Area of circle} - \text{area of ellipse} \\ &= \pi a^2 - \pi ab \\ &= \pi a(a - b)\end{aligned}$$

This is area of the ellipse whose semi axis are a and $(a - b)$.

One of the possible equations of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{(a - b)^2} = 1$

ILLUSTRATION 9.8

Let $f(x) = \text{Maximum} \{x^2, (1 - x)^2, 2x(1 - x)\}$, where $0 \leq x \leq 1$. Determine the area of the region bounded by the curves $y = f(x)$, x -axis, $x = 0$, and $x = 1$.

Sol. $f(x) = \text{Maximum} \{x^2, (1 - x)^2, 2x(1 - x)\}$

We draw the graphs of

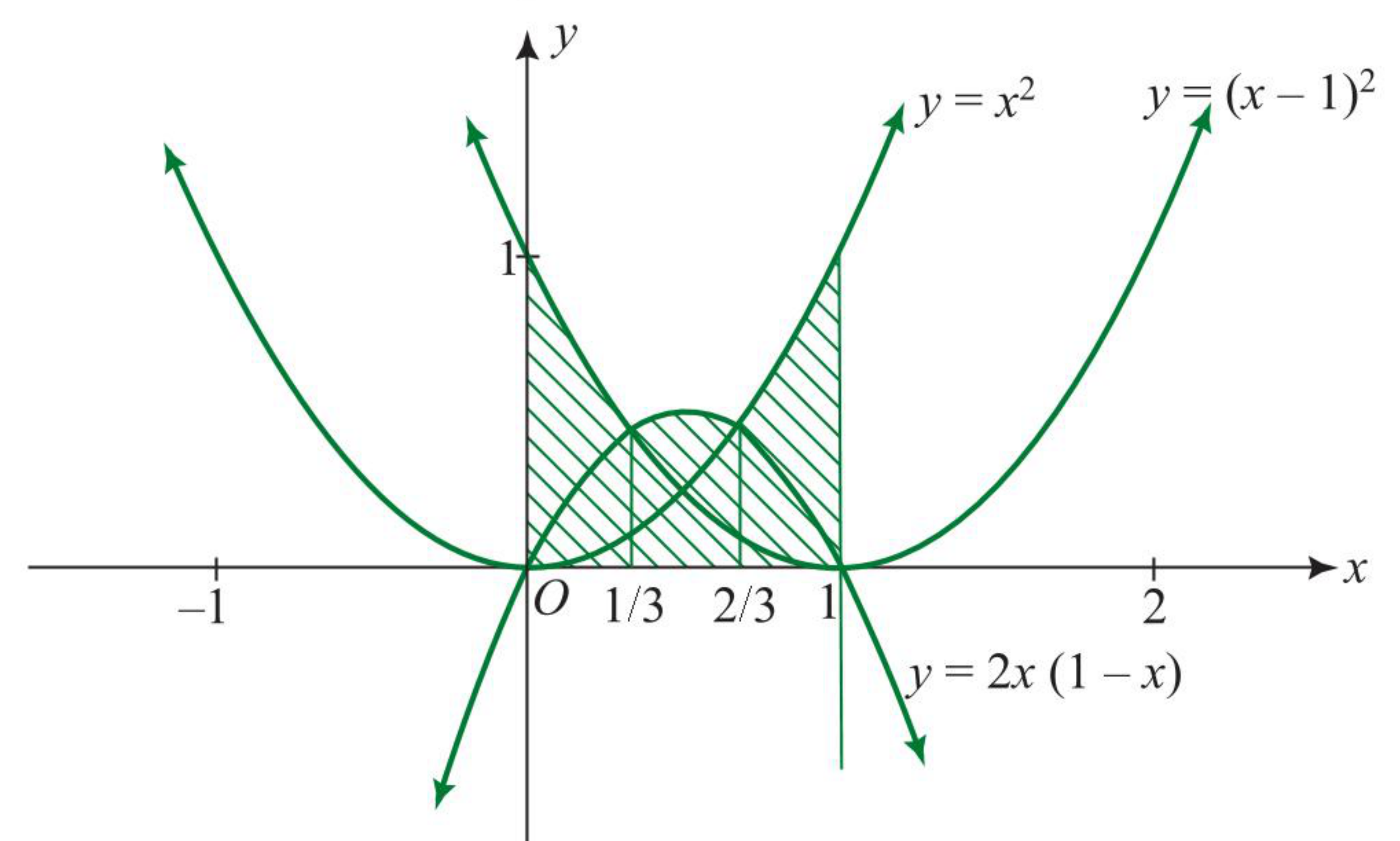
$$\begin{aligned}y &= x^2 & (1) \\ y &= (1 - x)^2 & (2) \\ y &= 2x(1 - x) & (3)\end{aligned}$$

Solving (1) and (3), we get $x^2 = 2x(1 - x)$

or $3x^2 = 2x \Rightarrow x = 0$ or $x = 2/3$.

Solving (2) and (3) we get $(1 - x)^2 = 2x(1 - x)$

$\Rightarrow x = 1/3$ and $x = 1$,



From the figure, it is clear that

$$f(x) = \begin{cases} (1-x)^2 & \text{for } 0 \leq x \leq 1/3 \\ 2x(1-x) & \text{for } 1/3 \leq x \leq 2/3 \\ x^2 & \text{for } 2/3 < x \leq 1 \end{cases}$$

The required area A is given by

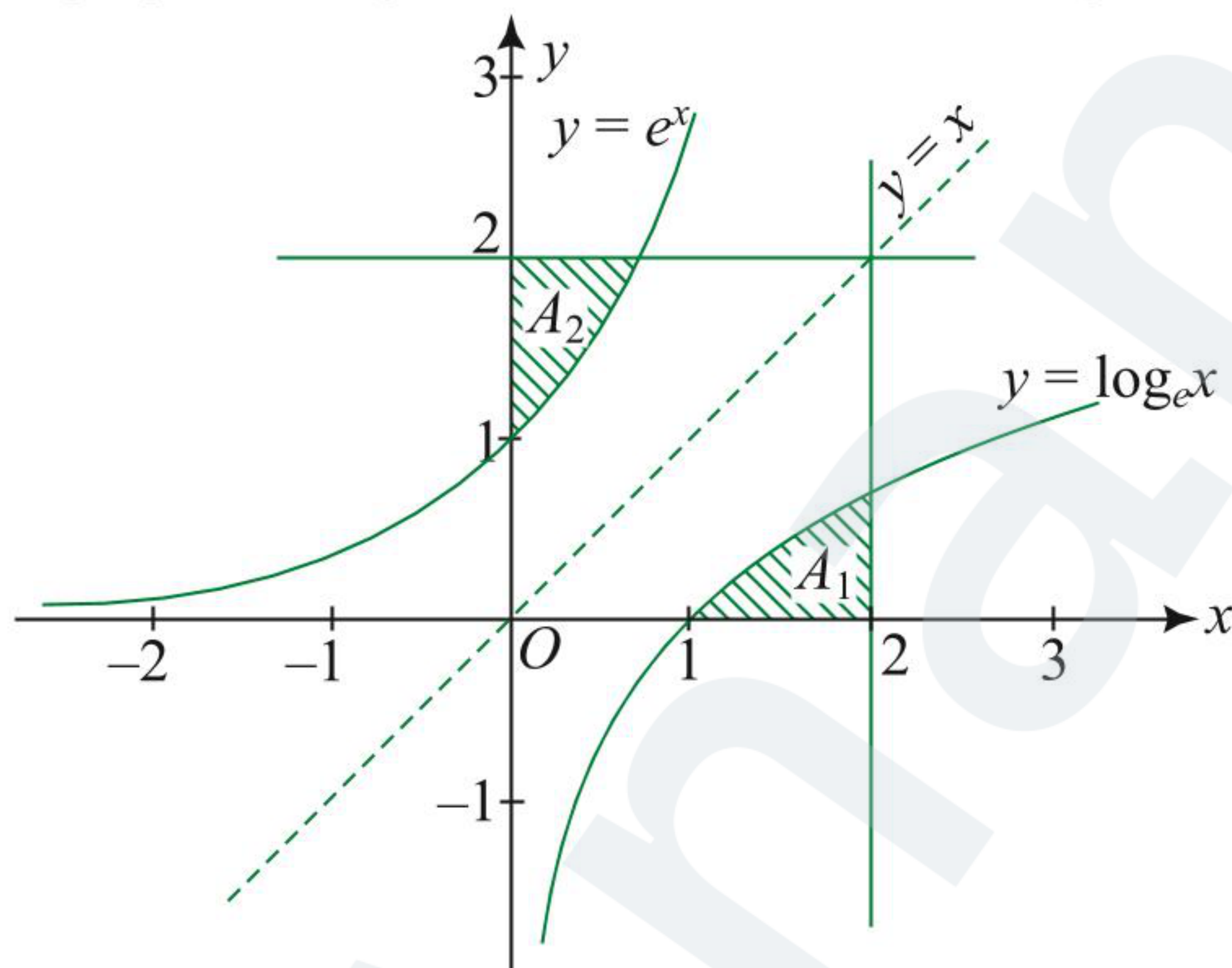
$$\begin{aligned} A &= \int_0^1 f(x) dx \\ &= \int_0^{1/3} (1-x)^2 dx + \int_{1/3}^{2/3} 2x(1-x) dx + \int_{2/3}^1 x^2 dx \\ &= \left[-\frac{1}{3}(1-x)^3 \right]_0^{1/3} + \left[\left(x^2 - \frac{2x^3}{3} \right) \right]_{1/3}^{2/3} + \left[\frac{x^3}{3} \right]_{2/3}^1 \\ &= -\frac{1}{3} \left(\frac{2}{3} \right)^3 + \frac{1}{3} + \left(\frac{2}{3} \right)^2 - \frac{2}{3} \left(\frac{2}{3} \right)^3 - \left(\frac{1}{3} \right)^2 + \frac{2}{3} \left(\frac{1}{3} \right)^3 \\ &\quad + \frac{1}{3} - \frac{1}{3} \left(\frac{2}{3} \right)^3 \\ &= \frac{17}{27} \text{ sq. units.} \end{aligned}$$

ILLUSTRATION 9.9

Consider the region formed by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$. Area enclosed by the curves $y = e^x$ and $y = \log_e x$, within this region is being removed. Then find the area of the remaining region

Sol. $y = e^x$ and $y = \log_e x$ are inverse to each other.

So, their graphs are symmetrical about the line $y = x$.



Area of the square $OABC$ is 4 sq. units.

Areas A_1 and A_2 are same.

So, area of the shaded region $= 2A_1$

$$\begin{aligned} &= 2 \int_1^2 \log_e x dx \\ &= 2 \left[x \log_e x - x \right]_1^2 \\ &= 2[2 \log_e 2 - 1] \text{ sq. unit} \end{aligned}$$

Therefore, the area of the unshaded region of the square

$$= 4 - 2[2 \log_e 2 - 1] = (6 - 4 \log_e 2) \text{ sq. units}$$

ILLUSTRATION 9.10

Draw a rough sketch of the curve, $y = \frac{x^2 + 3x + 2}{x^2 - 3x + 2}$ and find the area of the bounded region between the curve and x -axis.

Sol. $f(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$

$$f(x) = 0. \therefore x = -1, -2.$$

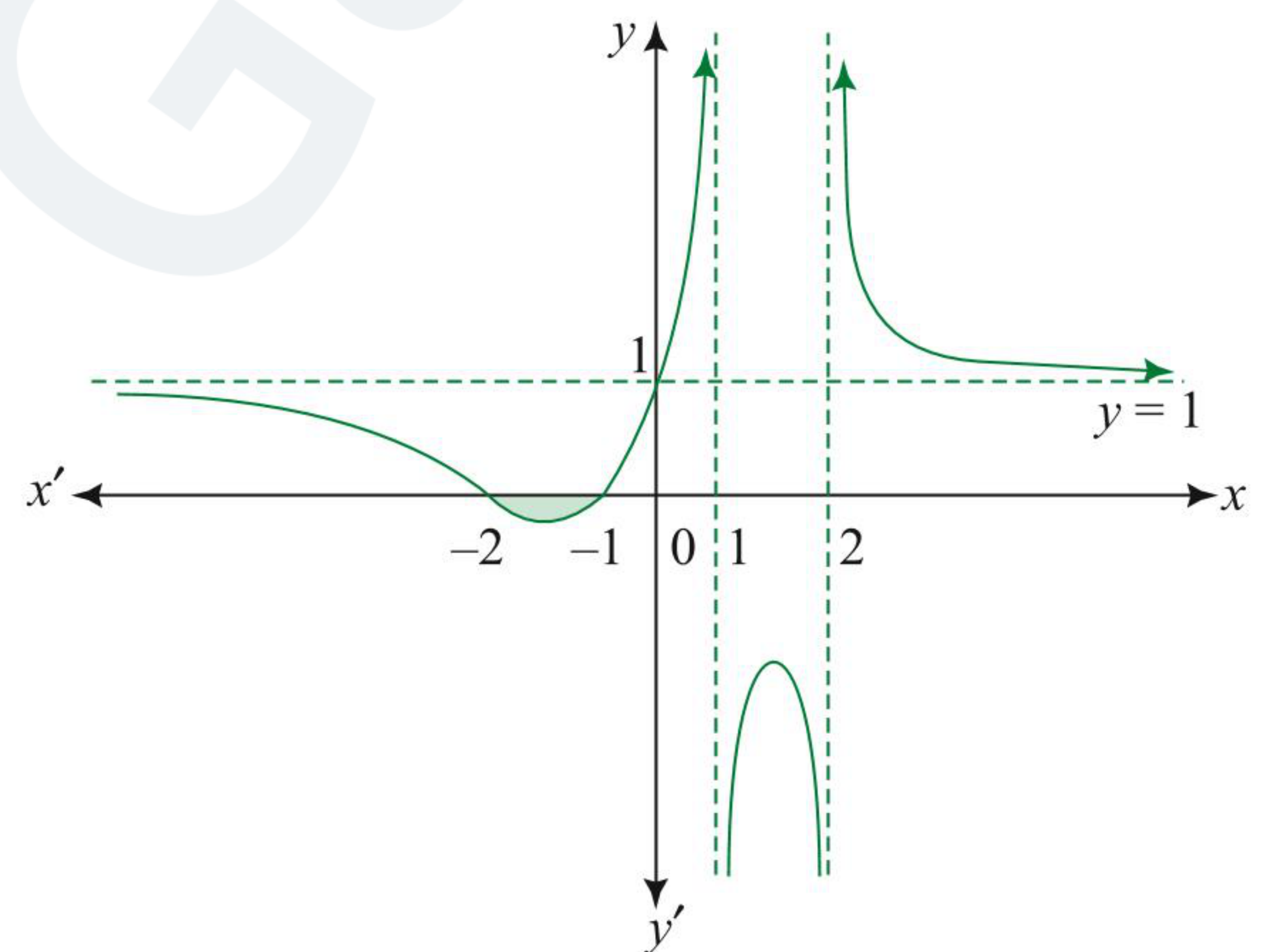
Also, $f(x)$ is discontinuous at $x = 1$ and $x = 2$.

$$\lim_{x \rightarrow 1^-} f(x) \rightarrow +\infty \text{ and } \lim_{x \rightarrow 1^+} f(x) \rightarrow -\infty$$

$$\lim_{x \rightarrow 2^-} f(x) \rightarrow -\infty \text{ and } \lim_{x \rightarrow 2^+} f(x) \rightarrow +\infty, f(0) = 1$$

$$\lim_{x \rightarrow \pm\infty} f(x) \rightarrow 1$$

So, graph of the function is as shown in the following figure.



The area of the shaded region is

$$\begin{aligned} \left| \int_{-2}^{-1} f(x) dx \right| &= \left| \int_{-2}^{-1} \left(\frac{x^2 + 3x + 2}{x^2 - 3x + 2} \right) dx \right| \\ &= \left| \int_{-2}^{-1} 1 + \frac{6x}{(x-1)(x-2)} dx \right| \\ &= \left| \int_{-2}^{-1} \left[1 + 6 \left(\frac{2}{x-2} - \frac{1}{x-1} \right) \right] dx \right| \\ &= \left| \left[x + 12 \log_e |x-2| - 6 \log_e |x-1| \right]_{-2}^{-1} \right| \\ &= \left| \left[1 + 12 \log_e \frac{3}{4} - 6 \log_e \frac{2}{3} \right] \right| \\ &= \left| \left[1 + 6 \log_e \frac{9}{16} - 6 \log_e \frac{2}{3} \right] \right| \\ &= \left| \left[1 + 6 \log_e \frac{27}{32} \right] \right| \\ &= 6 \log_e \frac{32}{27} - 1 \text{ sq. units} \end{aligned}$$

AREA BOUNDED BY CURVE AND AXIS WHEN GRAPH OF FUNCTION INTERSECTS x -AXIS

Let the graph of $y = f(x)$ for $x \in [a, b]$ intersect x -axis at $x = c$. If $f(x) \geq 0 \forall x \in [a, c]$ and $f(x) \leq 0 \forall x \in [c, b]$, then area bounded by curve and x -axis between lines $x = a$ and $x = b$ is

$$\int_a^b |f(x)| dx = \int_a^c f(x) dx - \int_c^b f(x) dx$$

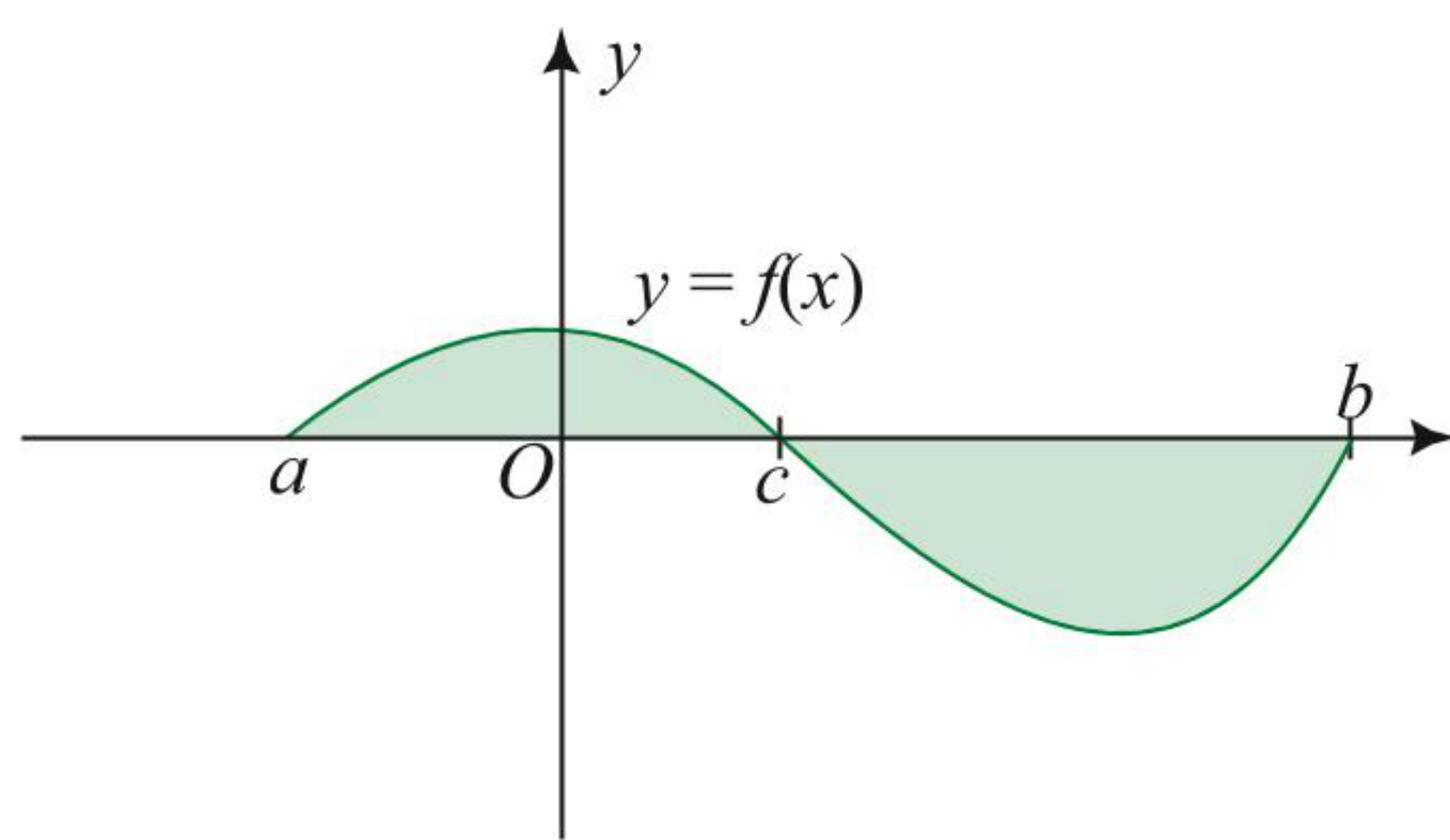
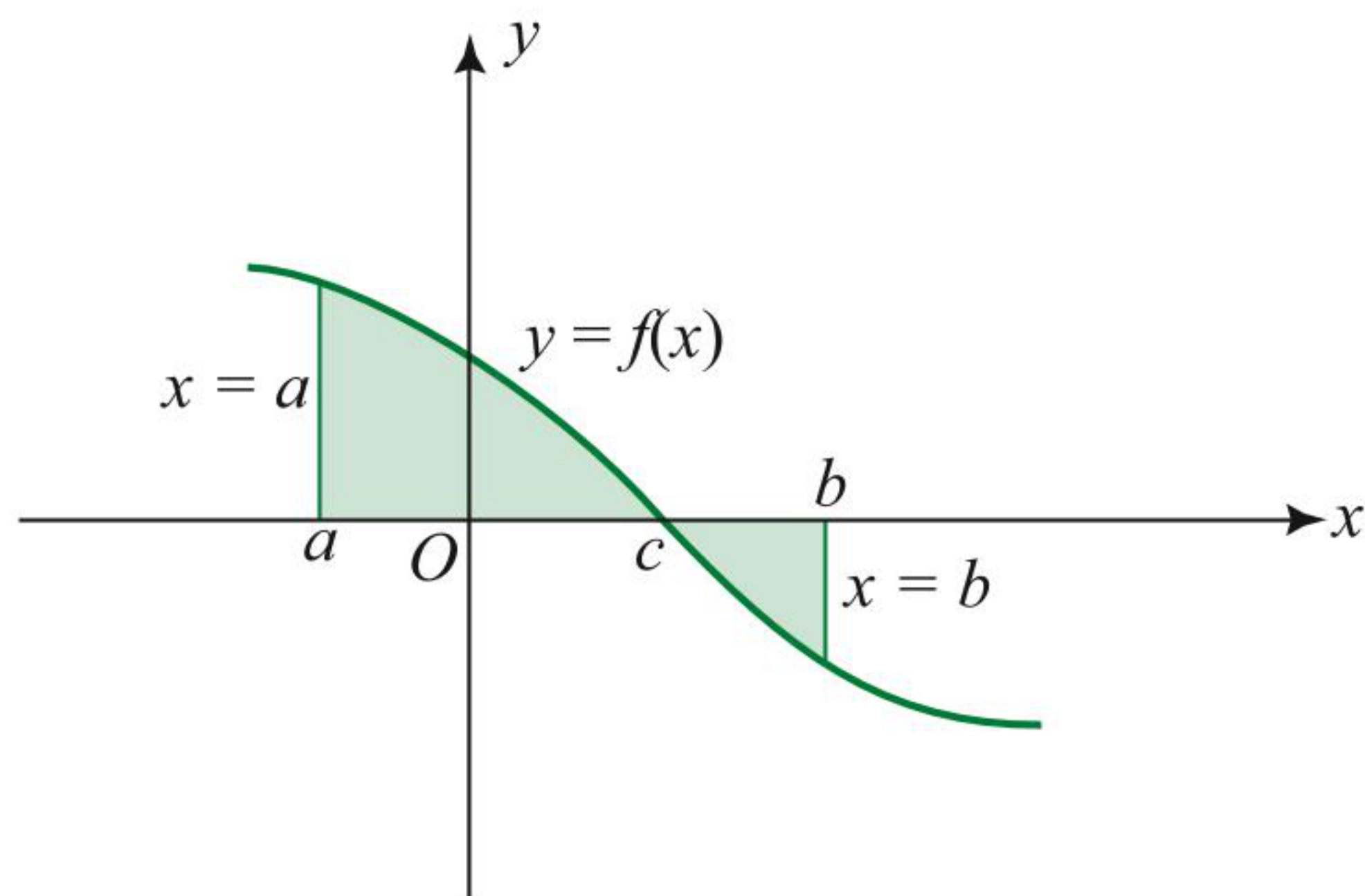
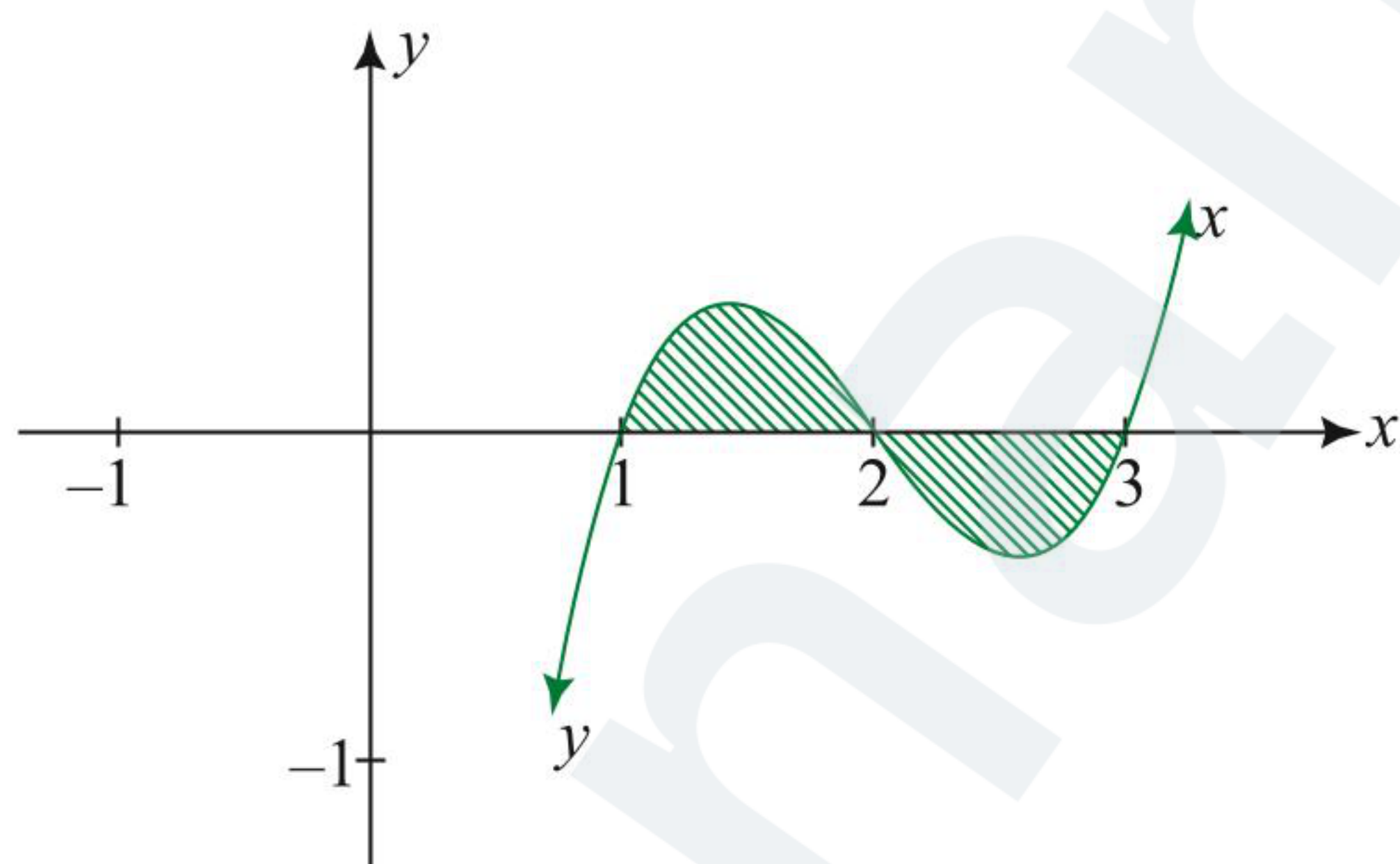


ILLUSTRATION 9.11

Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 1$ and $x = 3$.

Sol. $y = f(x) = (x - 1)(x - 2)(x - 3)$
 $y = 0$, then $x = 1, 2, 3$.

So, graph of the function is as shown in the following figure.



Since $f(2 - x) = -f(2 + x)$

$$\int_1^2 f(x) dx = \left| \int_2^3 f(x) dx \right|$$

$\therefore$ Area of the shaded region,

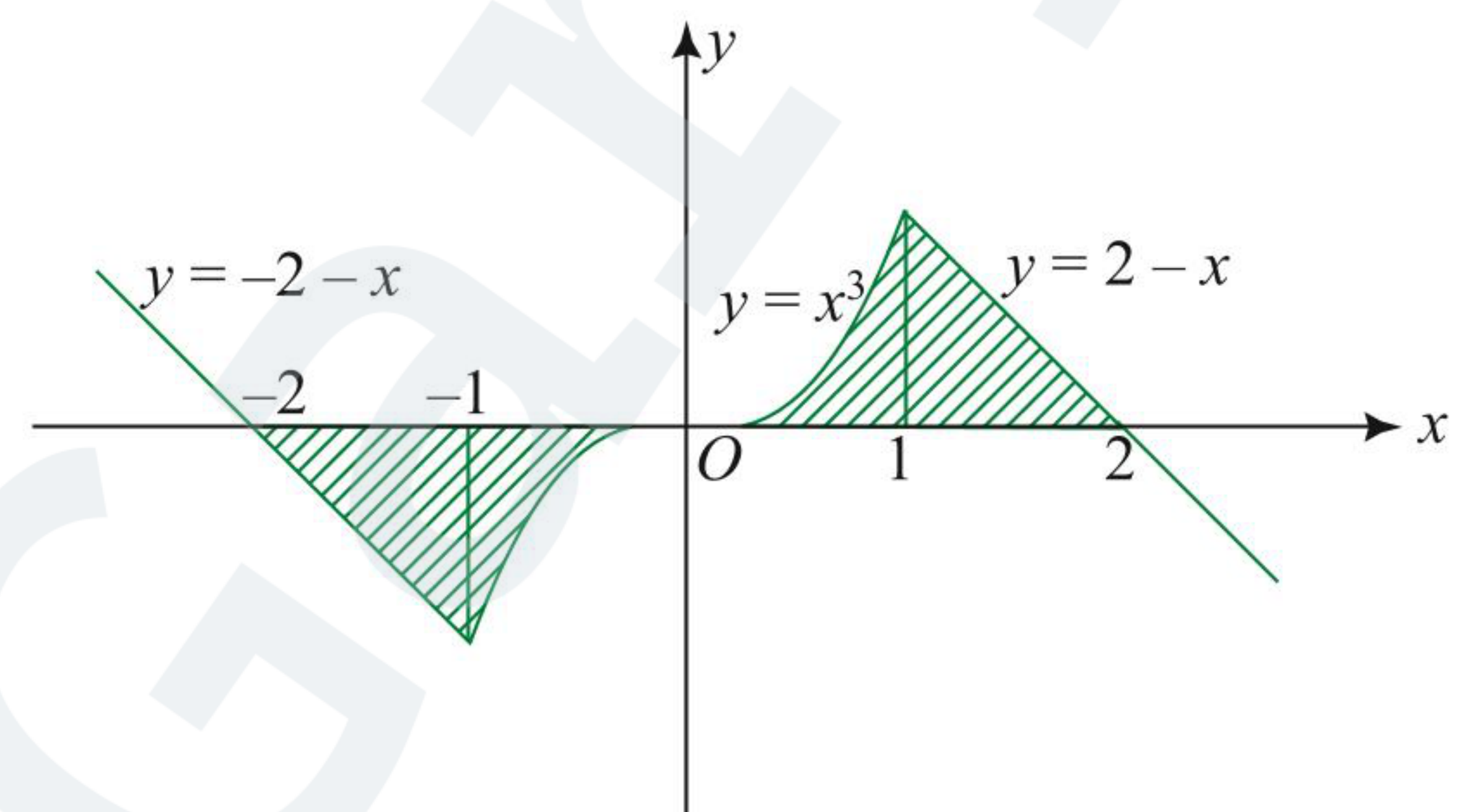
$$\begin{aligned} A &= 2 \int_1^2 (x^3 - 6x^2 + 11x - 6) dx \\ &= 2 \left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x \right]_1^2 \\ &= 2 \times \frac{1}{4} = \frac{1}{2} \end{aligned}$$

ILLUSTRATION 9.12

Find the area bounded by the curve $x = \begin{cases} -2 - y, & y < -1 \\ y^3, & -1 \leq y \leq 1 \\ 2 - y, & y > 1 \end{cases}$ and $x = 0$ is

Sol. Required area is same as area bounded by curve

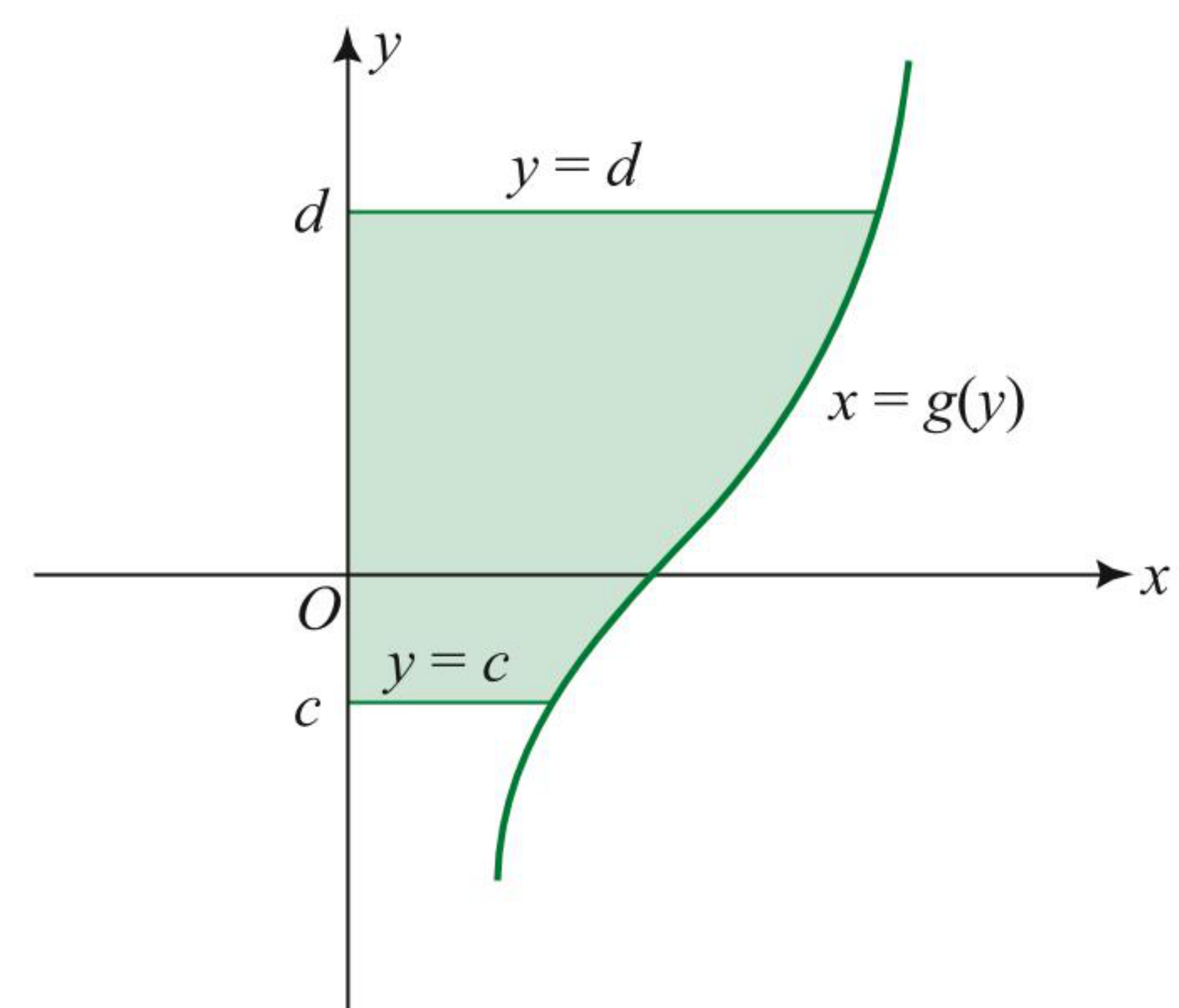
$$y = \begin{cases} -2 - x, & x < -1 \\ x^3, & -1 \leq x \leq 1 \text{ and } y = 0. \\ 2 - x, & x > 1 \end{cases}$$



$$\begin{aligned} \text{Required area} &= 2 \int_0^1 x^3 dx + 2 \int_1^2 (2 - x) dx \\ &= 2 \left[\frac{x^4}{4} \right]_0^1 + 2 \left[2x - \frac{x^2}{2} \right]_1^2 \\ &= 2 \left[\frac{1}{4} \right] + 2 \left[4 - \frac{4}{2} - 2 + \frac{1}{2} \right] \\ &= \frac{3}{2} \end{aligned}$$

AREA USING INTEGRATION ALONG y -AXIS

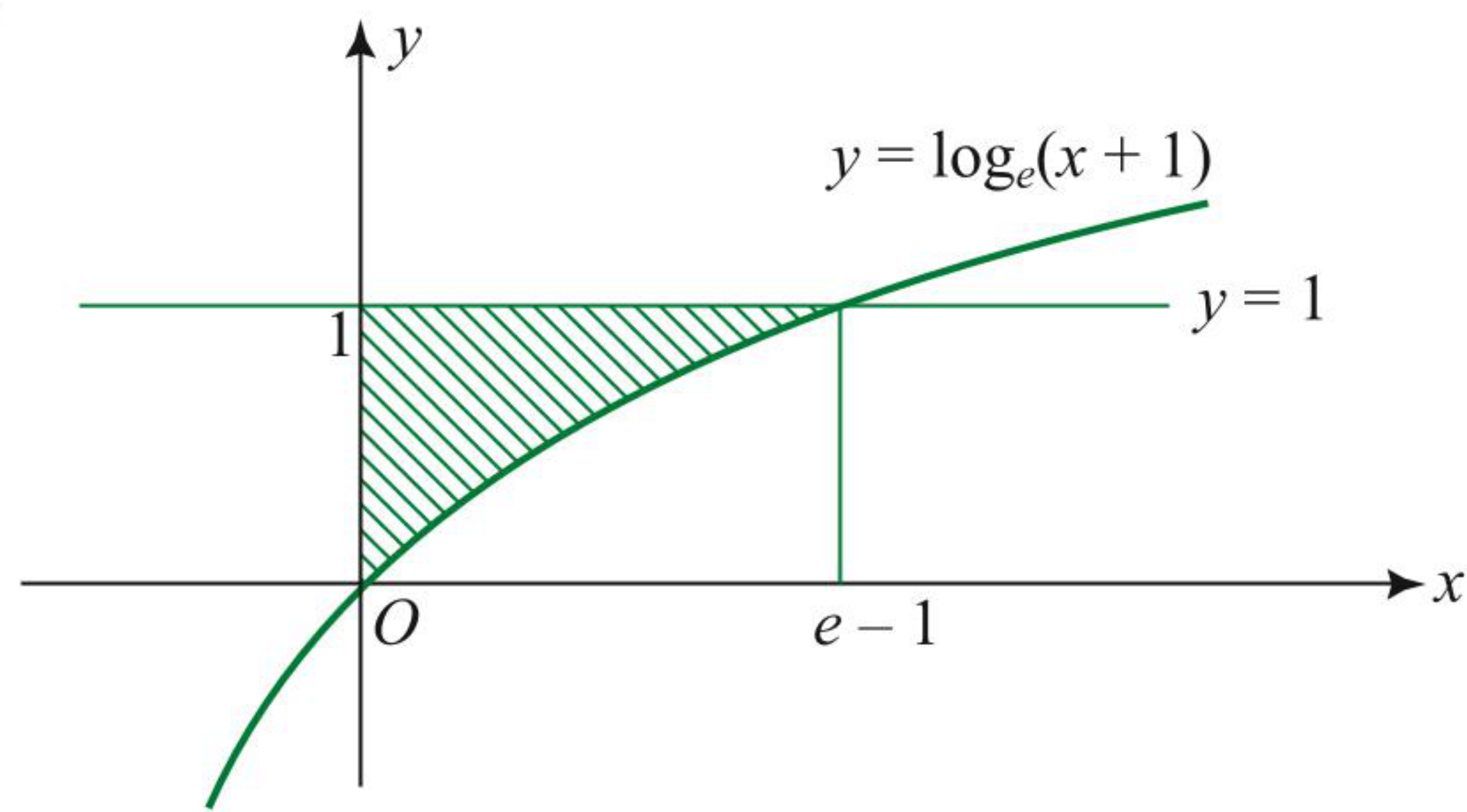
If we want to find the area bounded by curve $x = g(y)$ and y -axis between given limits of y as shown in the figure, we integrate function $x = g(y)$ along y -axis.



$$\text{So, area (of shaded region)} = \int_c^d x dy = \int_c^d g(y) dy.$$

ILLUSTRATION 9.13

Find the area enclosed by the graph of $y = \log_e(x + 1)$, y -axis, and the line $y = 1$

Sol.

$$y = \log_e(x+1)$$

$$\therefore x+1 = e^y$$

$$\Rightarrow x = e^y - 1$$

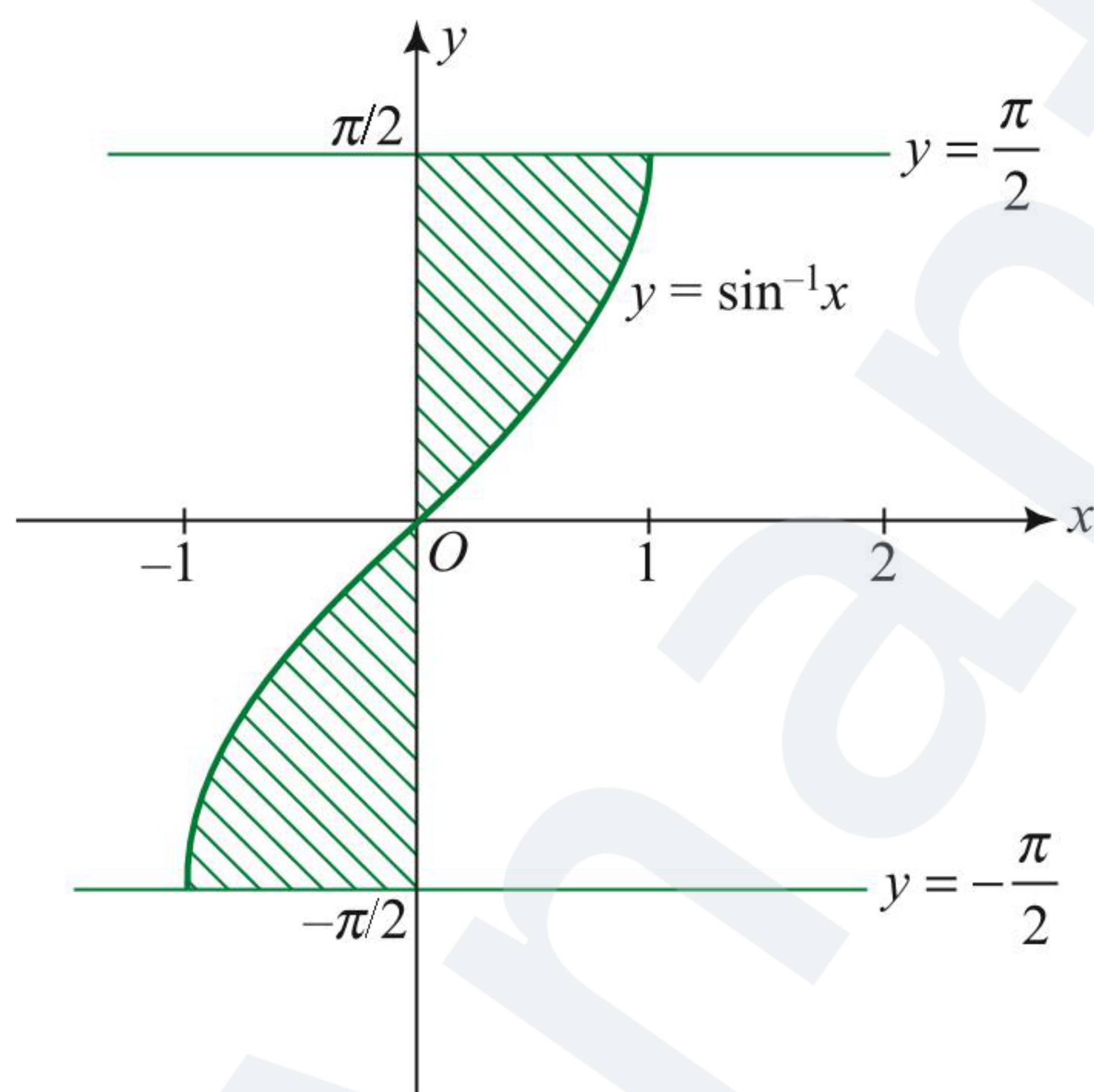
From the figure, required area (Intergrating along y-axis)

$$\begin{aligned} A &= \int_0^1 (e^y - 1) dy = [e^y - y]_0^1 \\ &= e^1 - 1 - 1 + 0 \\ &= e - 2 \end{aligned}$$

$$\begin{aligned} \text{Also, } A &= (e-1) \times 1 - \int_0^{e-1} \log_e(x+1) dx = (e-1) - \int_1^e \log_e x dx \\ &= (e-1) - 1 = e-2 \end{aligned}$$

ILLUSTRATION 9.14

Find the area bounded by the curve $y = \sin^{-1} x$ and the lines $x = 0$, $|y| = \frac{\pi}{2}$.

Sol.

The area of the shaded region (Intergrating along y-axis)

$$= 2 \int_{-\pi/2}^{\pi/2} |\sin y| dy = 2 \int_0^{\pi/2} \sin y dy = 2 \text{ sq. units}$$

CONCEPT APPLICATION EXERCISE 9.1

- Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
- Find the area enclosed by the curves $x^2 = y$, $y = x + 2$, and x -axis.

- A curve is given by $y = \begin{cases} \sqrt{4-x^2}, & 0 \leq x < 1 \\ \sqrt{3x}, & 1 \leq x \leq 3 \end{cases}$. Find the area lying between the curve and x -axis.

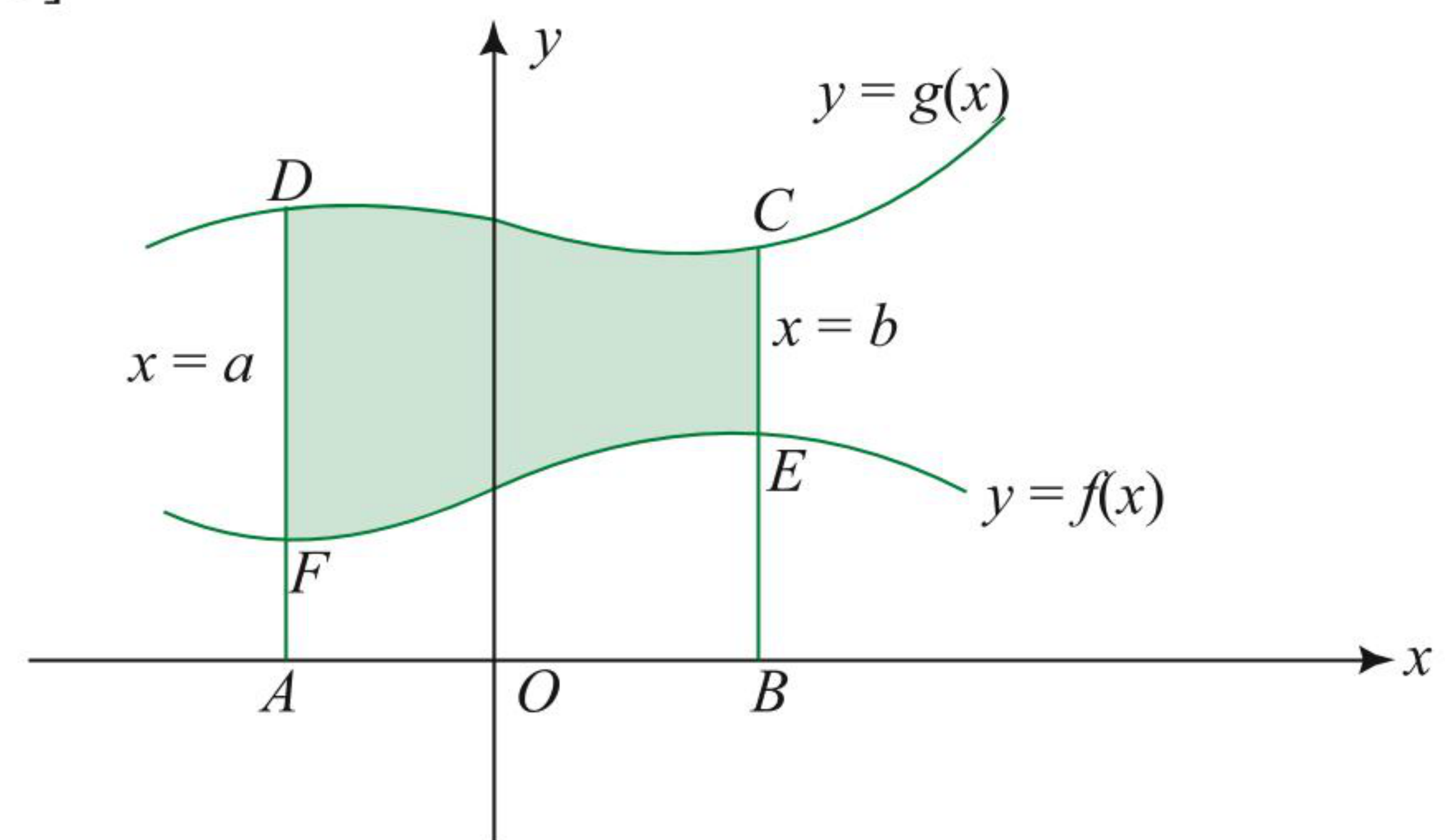
- Find the area bounded by $x = 2y - y^2$ and the y -axis.
- Find the area bounded by the x -axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$, and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, then find a .
- Find the area of the region bounded by the x -axis and the curves defined by $y = \tan x$ (where $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$) and $y = \cot x$ (where $\frac{\pi}{6} \leq x \leq \frac{3\pi}{2}$).
- Find the area bounded by $y = \left|\sin x - \frac{1}{2}\right|$ and $y = 1$ for $x \in [0, \pi]$.
- If the area bounded by the graph of $y = xe^{-ax}$ ($a > 0$) and the x -axis is $1/9$ then find the value of a .
- Find the area bounded by the curve $xy^2 = 4(2-x)$ and y -axis.

ANSWERS

- $\frac{a^2}{2} \left(\frac{\pi}{2} - 1\right)$ sq. units
- $\frac{5}{6}$ sq. units
- $\frac{1}{6} (2\pi - \sqrt{3} + 36)$ sq. units
- $\frac{4}{3}$ sq. units
- 4 sq. units; $a = 2\sqrt{2}$
- $\log \frac{3}{2}$ sq. units
- $\left(\frac{\pi}{2} + 2\right)$ sq. units
- 3
- 4π sq. units

AREA BOUNDED BY TWO CURVES**AREA BOUNDED BY CURVES IN GIVEN INTERVAL**

Let us find the area bounded by the straight lines $x = a$, $x = b$ ($a < b$) and the curves $y = f(x)$ and $y = g(x)$, provided $f(x) \leq g(x)$ for all $x \in [a, b]$



Clearly from the figure, area of the shaded region

$$= \text{Area of the region } ABCD - \text{Area of the region } ABEF$$

$$= \int_a^b g(x) dx - \int_a^b f(x) dx = \int_a^b \left(\underbrace{g(x)}_{\text{Upper curve}} - \underbrace{f(x)}_{\text{Lower curve}} \right) dx$$

Note:

- Area bounded by ellipse whose semi-axes are a and b is πab sq. units.
- Area bounded by parabola $y^2 = 4ax$ and $x^2 = 4by$ is $\left| \frac{16ab}{3} \right|$ sq. units.

ILLUSTRATION 9.15

Find the area of the region bounded by the curves $y = \sqrt{x+2}$ and $y = \frac{1}{x+1}$ between the lines $x = 0$ and $x = 2$.

Sol. $y = \sqrt{x+2}$... (1)

$\therefore y^2 = x+2$

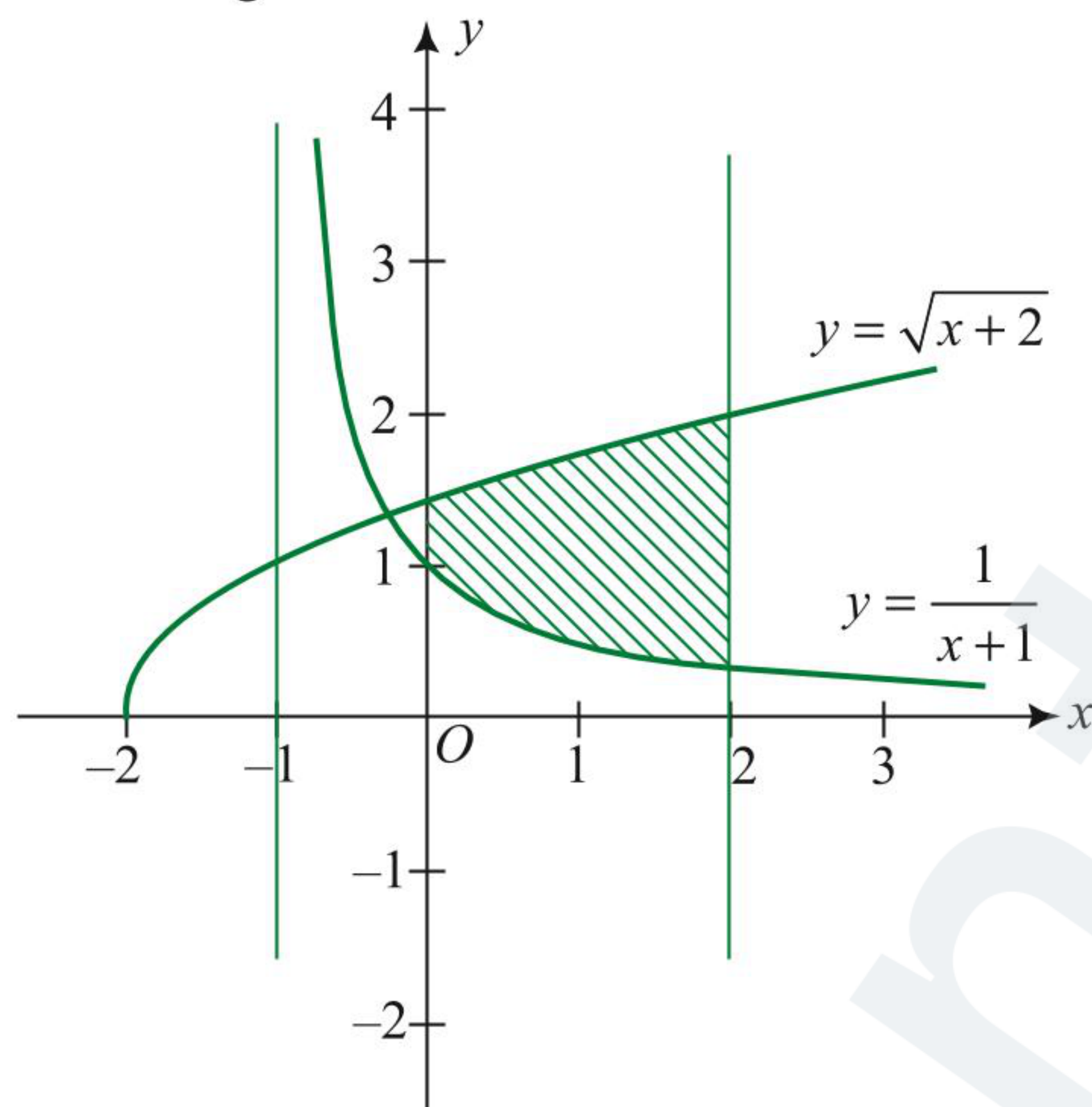
So, curve (1) is parabola for $y \geq 0$.

$y = \frac{1}{x+1}$... (2)

$\lim_{x \rightarrow -1^+} \frac{1}{x+1} = \infty$ and $\lim_{x \rightarrow \infty} \frac{1}{x+1} = 0$

$y(0) = 1$.

Also, y is decreasing function.



From the figure required area is

$$\begin{aligned} A &= \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1} \right) dx \\ &= \left[\frac{2}{3} (x+2)^{3/2} - \log_e (x+1) \right]_0^2 \\ &= \left(\frac{2}{3} \times 8 - \log_e 3 \right) - \left(\frac{2}{3} \times 2^{3/2} - \log_e 1 \right) \\ &= \frac{16 - 4\sqrt{2}}{3} - \log_e 3 \end{aligned}$$

ILLUSTRATION 9.16

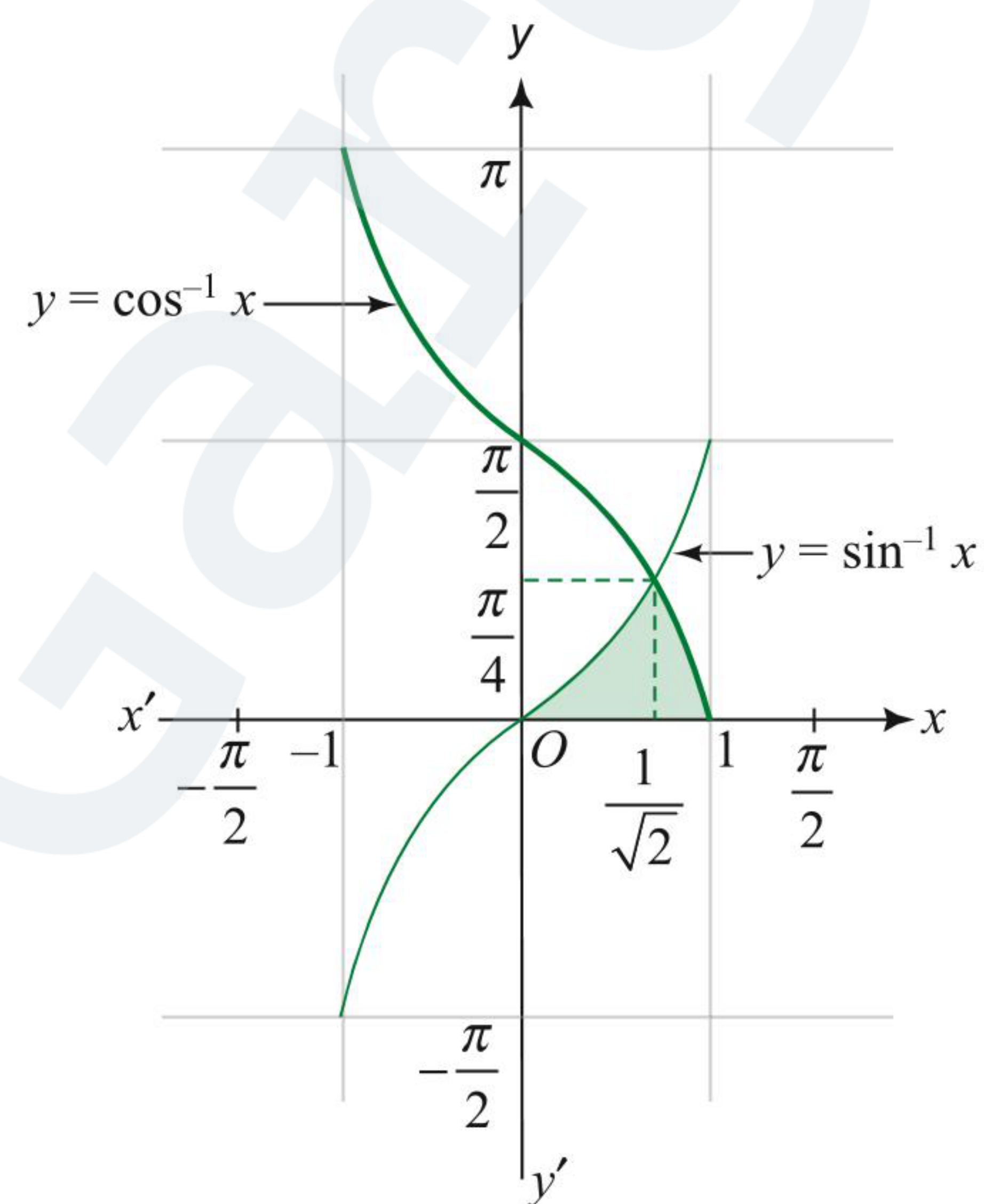
Find the area bounded by $y = \sin^{-1} x$, $y = \cos^{-1} x$, and the x -axis.

Sol. We have to find the area bounded by $y = \sin^{-1} x$, $y = \cos^{-1} x$, and the x -axis. If integration along x -axis is used then

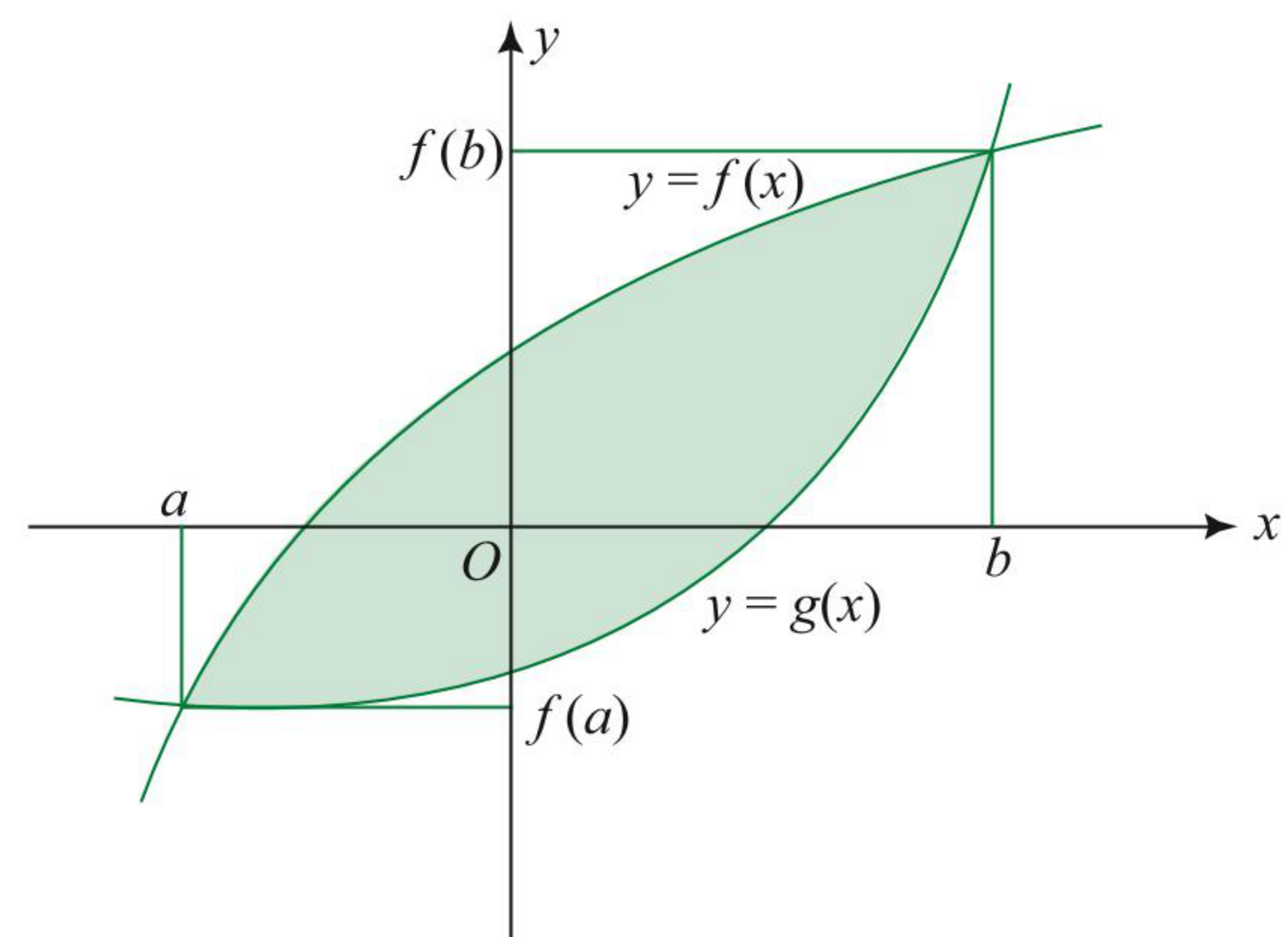
$$A = \int_0^{1/\sqrt{2}} \sin^{-1} x \, dx + \int_{1/\sqrt{2}}^1 \cos^{-1} x \, dx$$

If integration along y -axis is used, then

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos y - \sin y) \, dy \\ &= [\sin y + \cos y]_0^{\pi/4} \\ &= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right] \\ &= \sqrt{2} - 1 \end{aligned}$$

**AREA BOUNDED BY TWO CURVES WHEN CURVES INTERSECT AT TWO POINTS**

Consider the following graphs of the functions $y = f(x)$ and $y = g(x)$ which intersect at two points where $x = a$ and $x = b$.



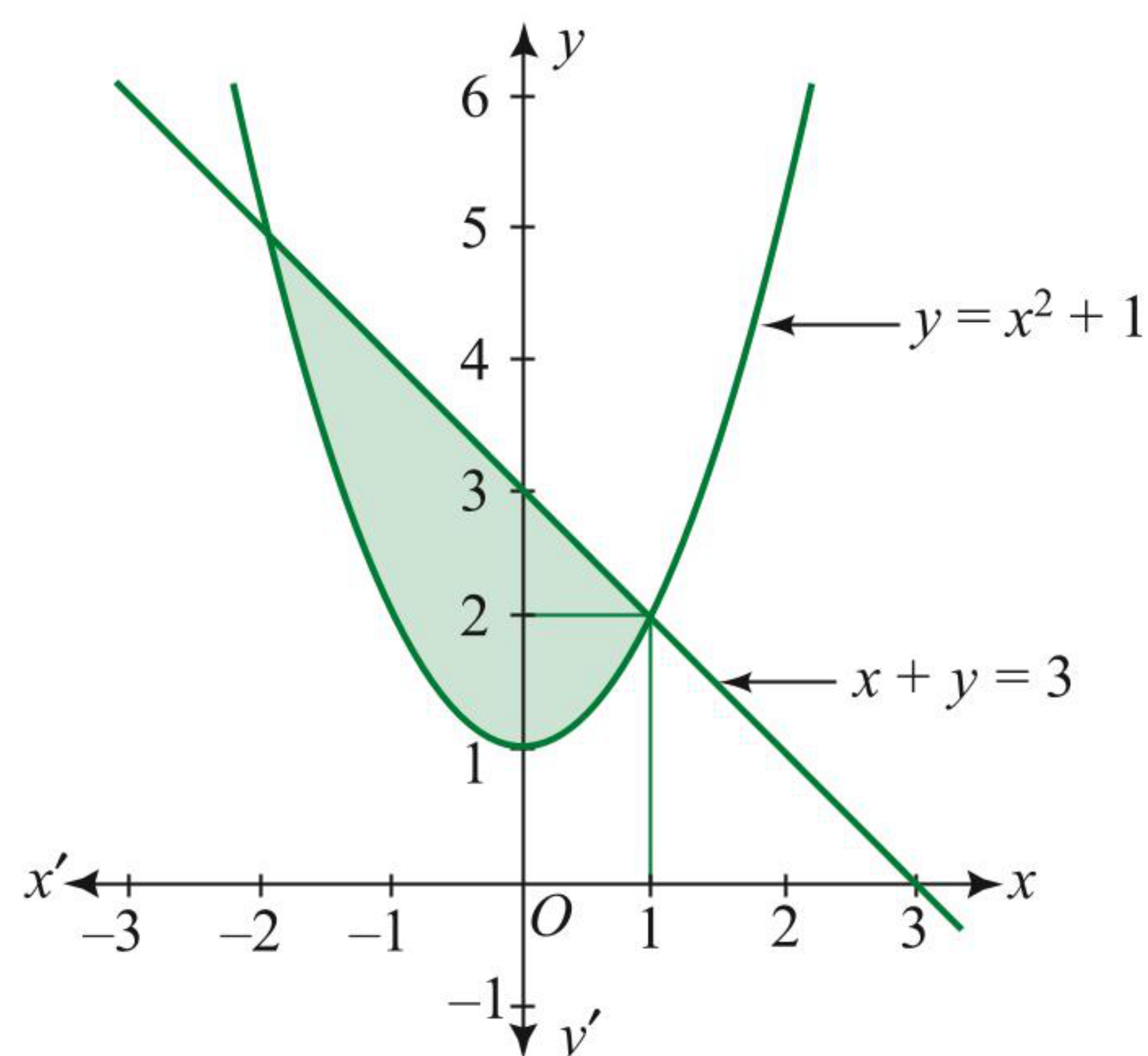
Area bounded by curves $= \int_a^b [f(x) - g(x)] dx$, where a and b are roots of the equation $f(x) = g(x)$

Also, area $= \int_{f(a)}^{f(b)} [g^{-1}(y) - f^{-1}(y)] dy$

if we choose to integrate along y -axis.

ILLUSTRATION 9.17

Find the area bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.

Sol.

The two curves meet at points where $3 - x = x^2 + 1$,

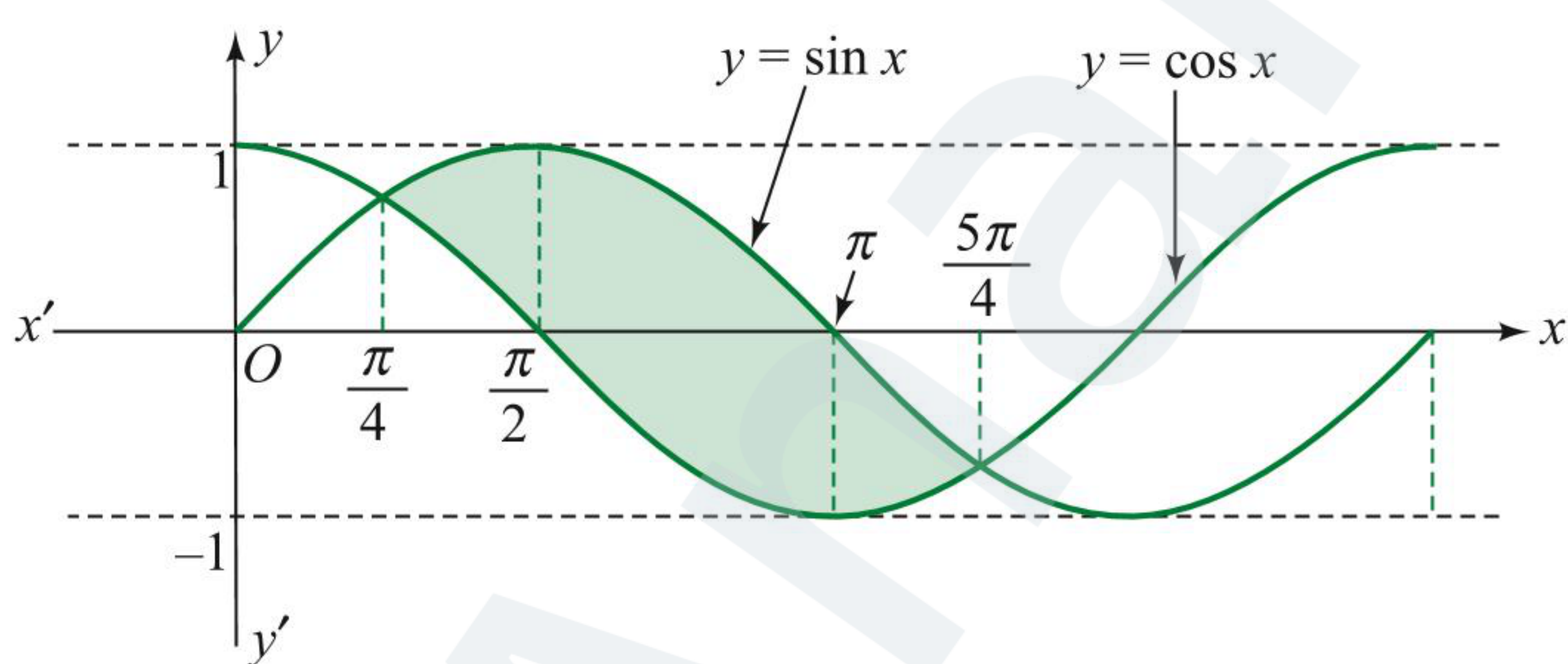
i.e., $x^2 + x - 2 = 0$

or $(x + 2)(x - 1) = 0$ or $x = -2, 1$

$$\begin{aligned} \therefore \text{Required area} &= \int_{-2}^1 [(3-x) - (x^2 + 1)] dx \\ &= \int_{-2}^1 (2 - x - x^2) dx \\ &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - \frac{4}{2} + \frac{8}{3} \right) \\ &= \frac{9}{2} \text{ sq. units} \end{aligned}$$

ILLUSTRATION 9.18

Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ between two consecutive points of the intersection.

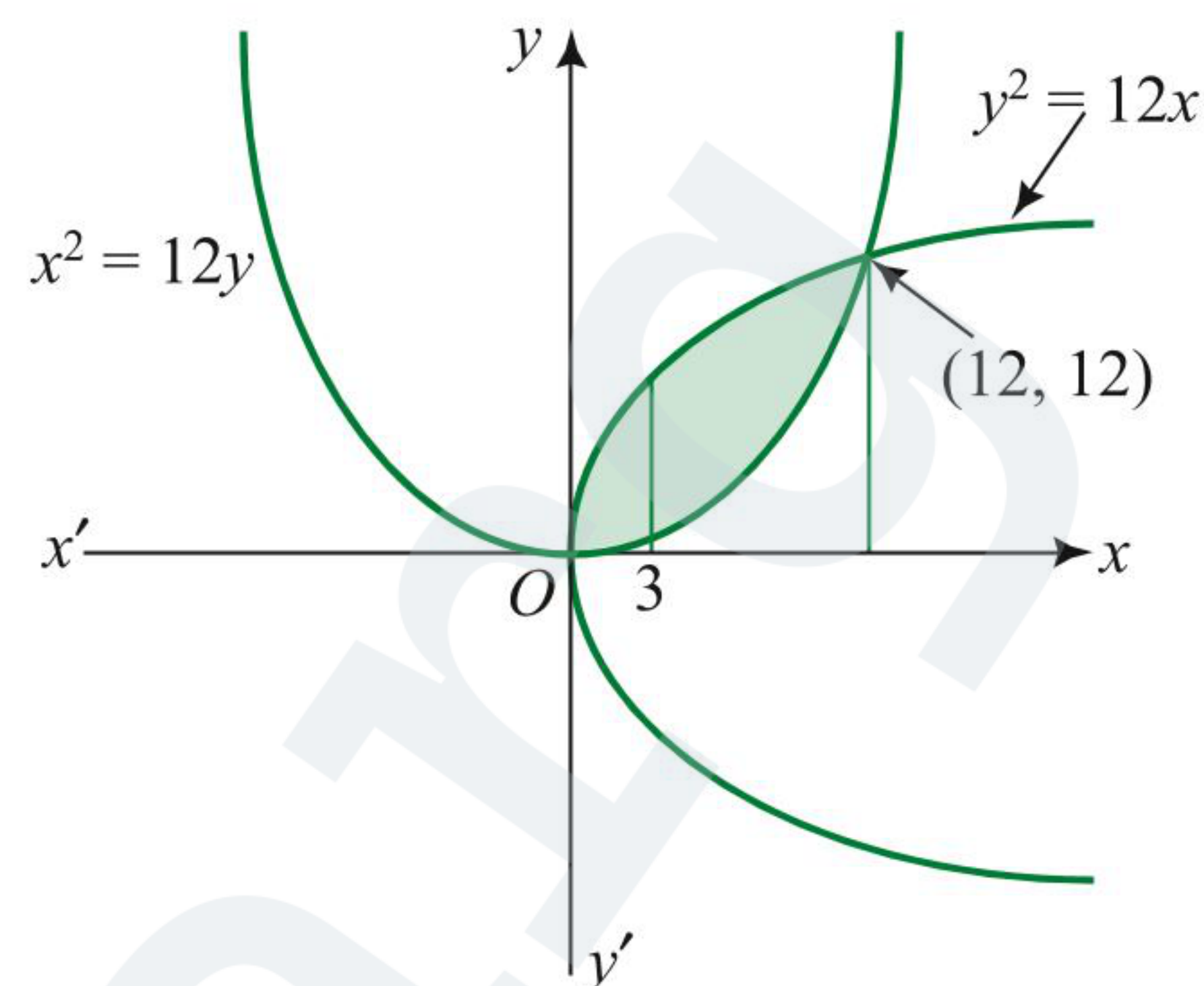
Sol.

Two consecutive points of intersection of $y = \sin x$ and $y = \cos x$ can be taken as $x = \pi/4$ and $x = 5\pi/4$. Therefore,

$$\begin{aligned} \text{Required area} &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx \\ &= [-\cos x - \sin x]_{\pi/4}^{5\pi/4} \\ &= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ &= 2\sqrt{2} \text{ sq. units} \end{aligned}$$

ILLUSTRATION 9.19

Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

Sol.

A_1 = Area bounded by $y^2 = 12x$, $x^2 = 12y$, and line $x = 3$

$$\begin{aligned} &= \int_0^3 \sqrt{12x} dx - \int_0^3 \frac{x^2}{12} dx \\ &= \sqrt{12} \left[\frac{2x^{3/2}}{3} \right]_0^3 - \left[\frac{x^3}{36} \right]_0^3 = \frac{45}{4} \text{ sq. units.} \end{aligned}$$

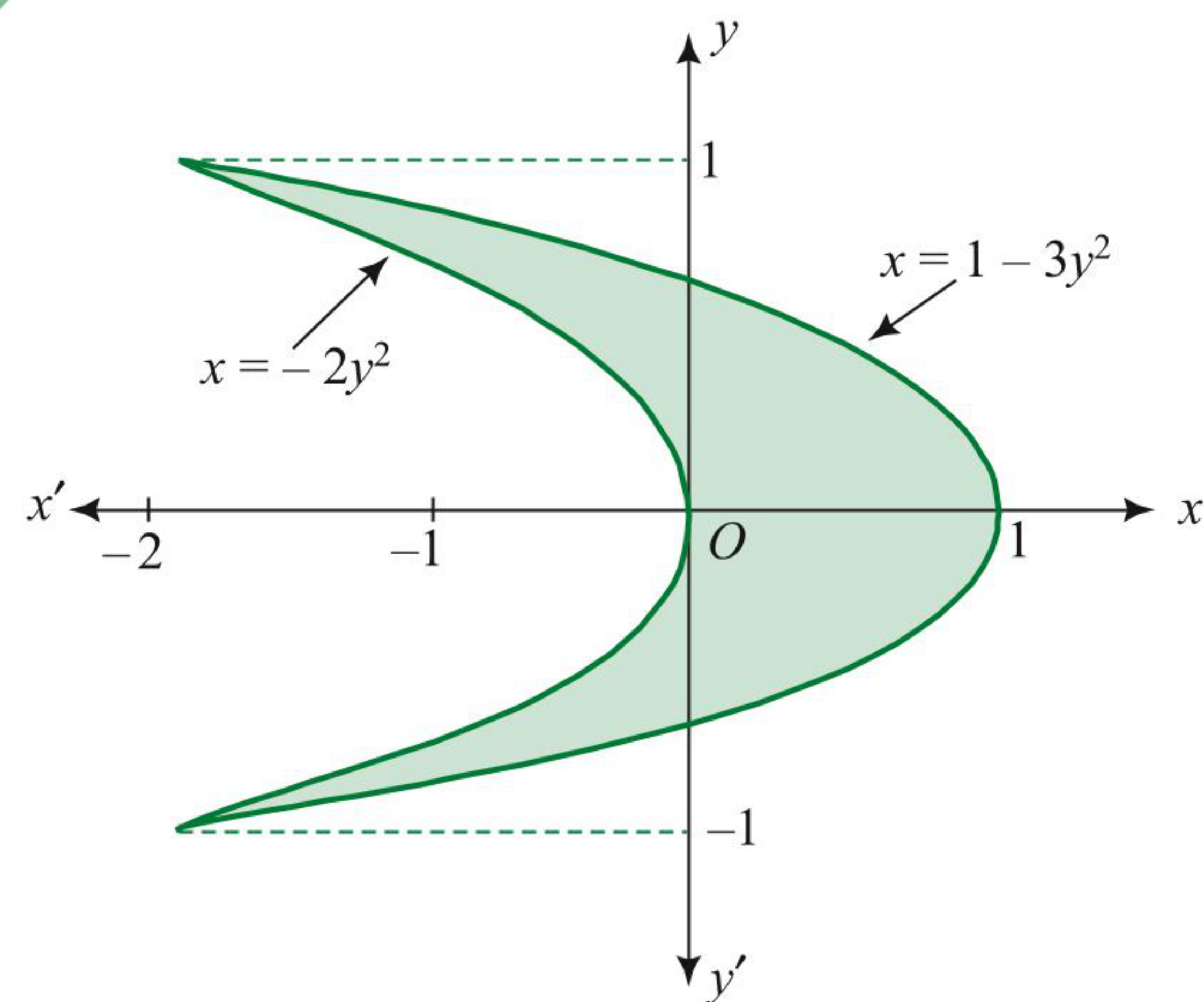
A_2 = Area bounded by $y^2 = 12x$ and $x^2 = 12y$

$$= \frac{16(3)(3)}{3} = 48 \text{ sq. units}$$

$$\therefore \text{Required ratio} = \frac{\frac{45}{4}}{48 - \frac{45}{4}} = \frac{45}{147}$$

ILLUSTRATION 9.20

Find the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Sol.

Solving the equation $x = -2y^2$, $x = 1 - 3y^2$, we find that the ordinates of the point of intersection of the two curves are

$$y_1 = -1, y_2 = 1. \text{ The points are } (-2, -1) \text{ and } (-2, 1).$$

The required area is given by (Integrating along y-axis)

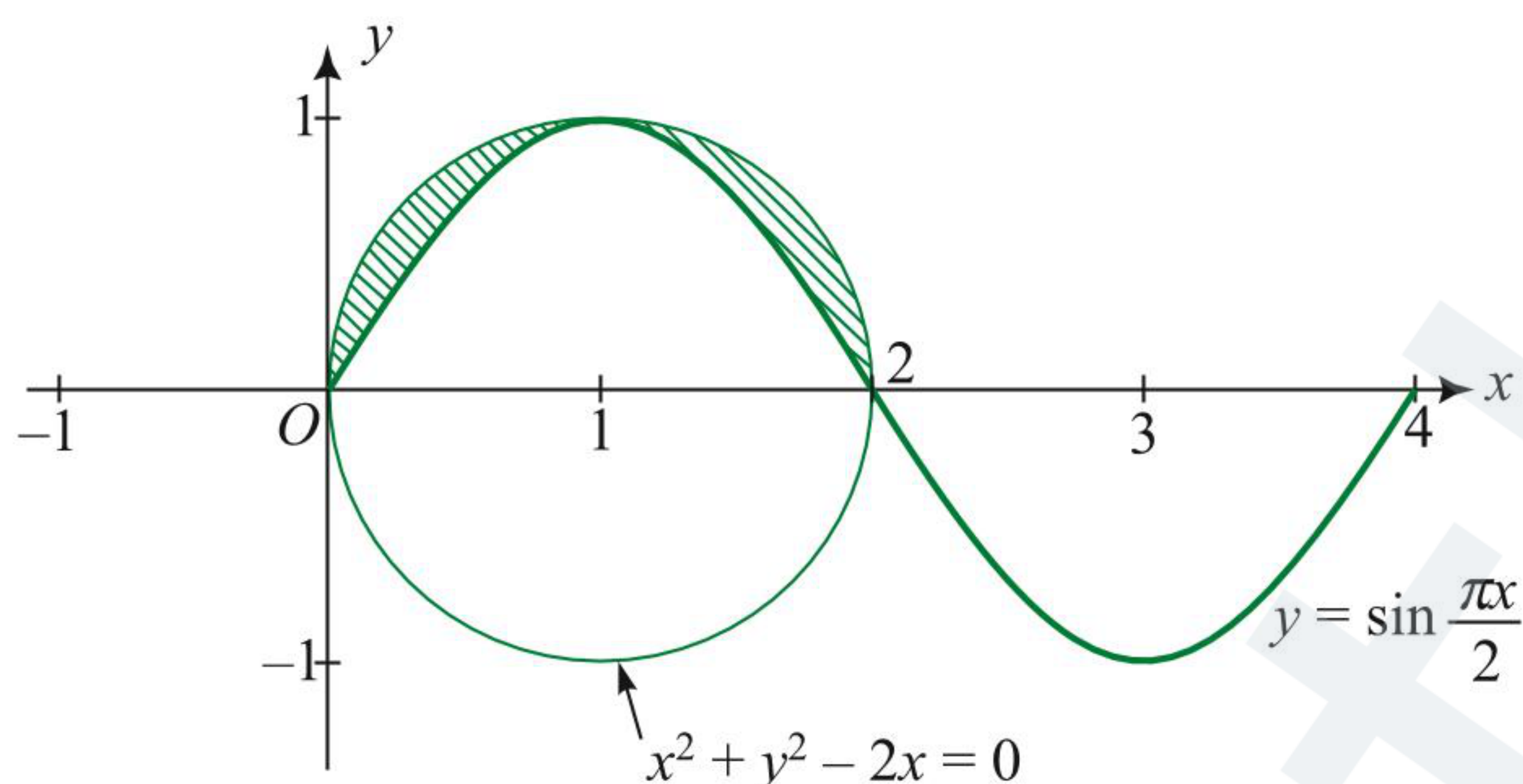
$$\begin{aligned}
 A &= 2 \int_0^1 (x_1 - x_2) dy \\
 &= 2 \int_0^1 [(1 - 3y^2) - (-2y^2)] dy \\
 &= 2 \int_0^1 (1 - y^2) dy \\
 &= 2 \left[y - \frac{y^3}{3} \right]_0^1 = \frac{4}{3}
 \end{aligned}$$

ILLUSTRATION 9.21

The area common to regions $x^2 + y^2 - 2x \leq 0$ and $y \geq \sin \frac{\pi x}{2}$.

Sol. For $x^2 + y^2 - 2x \leq 0$
 points lie on or inside circle $(x - 1)^2 + y^2 = 1$
 For $y \geq \sin \frac{\pi x}{2}$, points lie on or above $y = \sin \frac{\pi x}{2}$.
 $y = \sin \frac{\pi x}{2}$ has period 4.

The graphs of curves and the required region is as shown in the following figure.



From the figure, required area = area of semicircle - $\int_0^2 \sin \frac{\pi x}{2} dx$

$$\begin{aligned}
 &= \frac{\pi}{2} + \frac{2}{\pi} \left[\cos \frac{\pi x}{2} \right]_0^2 \\
 &= \frac{\pi}{2} - \frac{4}{\pi}
 \end{aligned}$$

ILLUSTRATION 9.22

Find the area of the region enclosed by the curves $y = x \log_e x$ and $y = 2x - 2x^2$.

Sol. Curve tracing, $y = x \log_e x$

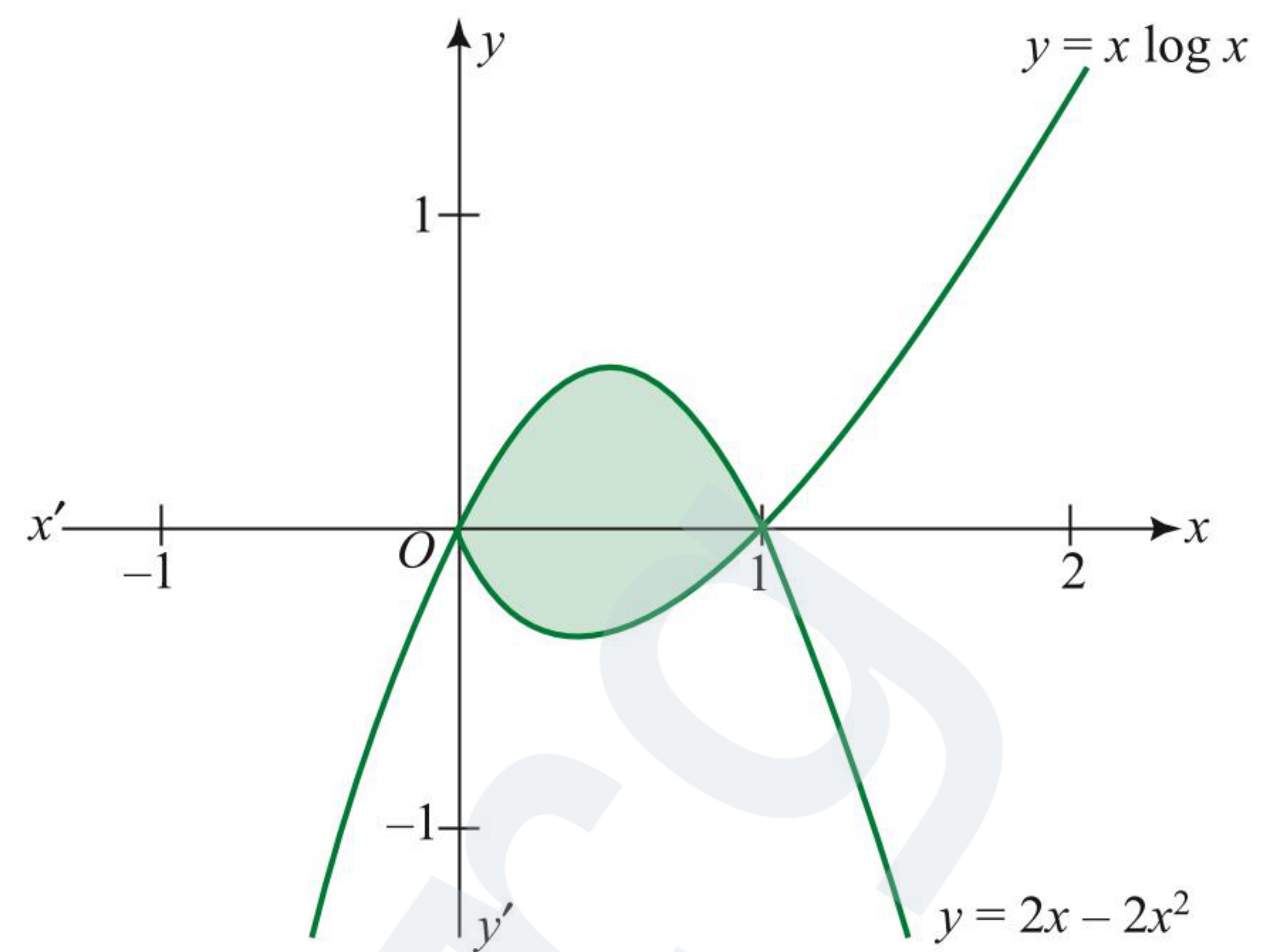
Clearly $x > 0$,

For $0 < x < 1$, $x \log_e x < 0$, and for $x > 1$, $x \log_e x > 0$

Also $x \log_e x = 0$ for $x = 1$

Further $\frac{dy}{dx} = 0 \Rightarrow 1 + \log_e x = 0$ or $x = 1/e$, which is point of minima.

$\lim_{x \rightarrow \infty} x \log_e x = \infty$. Graph of function is as shown in the figure.
 $y = 2x - 2x^2$ is downward parabola intersecting x -axis at $x = 0$ and $x = 1$.



$$\begin{aligned}
 \text{Required area} &= \int_0^1 (2x - 2x^2) dx - \int_0^1 x \log x dx \\
 &= \left[x^2 - \frac{2x^3}{3} \right]_0^1 - \left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right]_0^1 \\
 &= \left(1 - \frac{2}{3} \right) - \left[0 - \frac{1}{4} - \frac{1}{2} \lim_{x \rightarrow 0} x^2 \log x \right] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}
 \end{aligned}$$

ILLUSTRATION 9.23

Find the area bounded by $y^2 \leq 4x$, $x^2 + y^2 \geq 2x$, and $x \leq y + 2$ in the first quadrant.

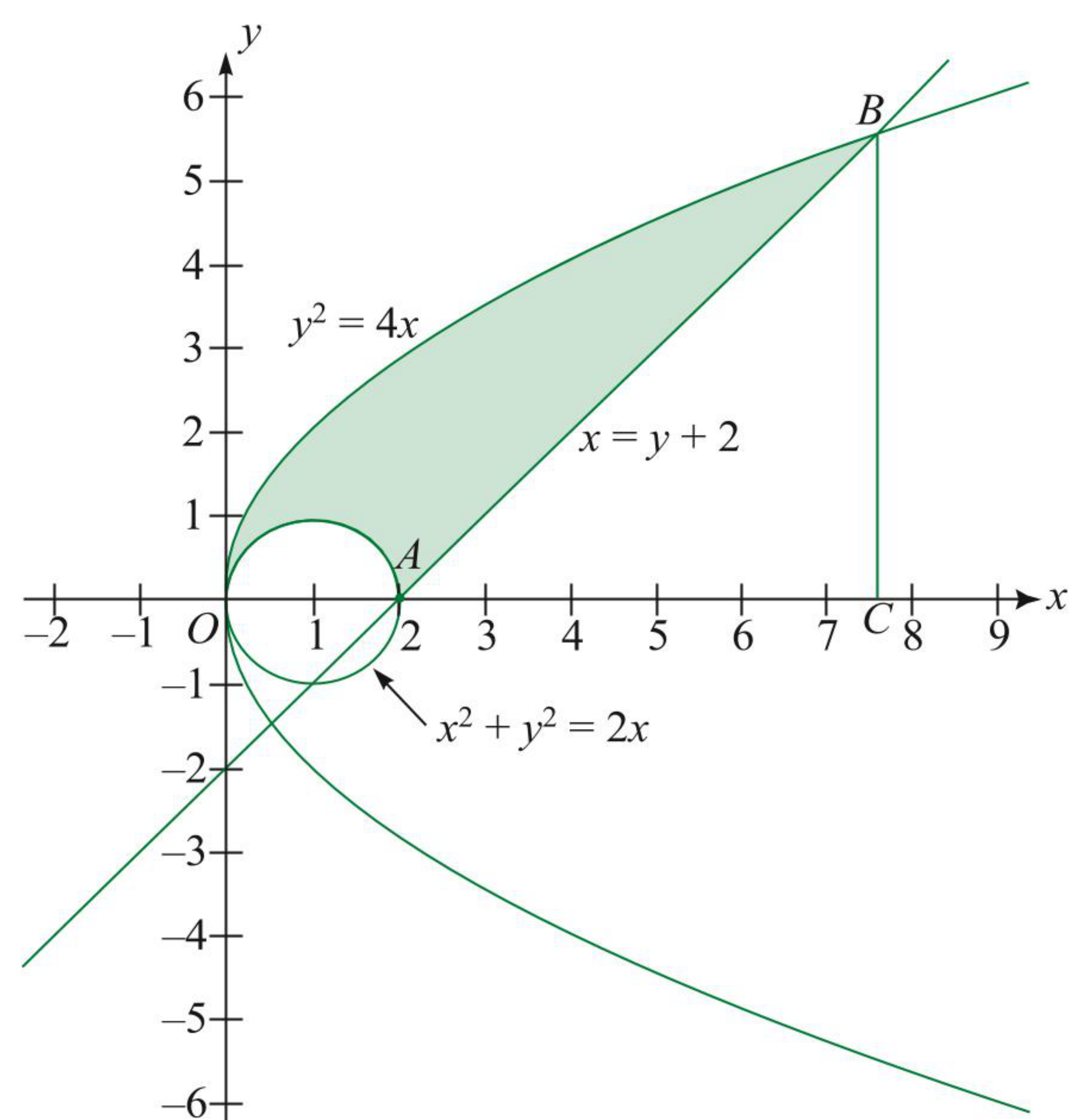
Sol. For $y^2 \leq 4x$, points lie on or inside parabola $y^2 = 4x$
 For $x^2 + y^2 \geq 2x$, points lie on or outside circle $(x - 1)^2 + (y - 0)^2 = 1$
 For $x \leq y + 2$, points lie on or to the left side (towards origin) of the line $x = y + 2$.

The graphs of the curves is as shown in the figure.

Solving parabola and line $y = x + 2$, we get

$$\begin{aligned}
 (x - 2)^2 &= 4x \\
 \text{or } x^2 - 8x + 4 &= 0
 \end{aligned}$$

$$\text{or } x = \frac{8 \pm \sqrt{48}}{2} = 2(2 \pm \sqrt{3}) = (1 \pm \sqrt{3})^2$$



From the figure, required area is

$$\begin{aligned}
 A &= \int_0^{(\sqrt{3}+1)^2} \sqrt{4x} \, dx - \text{Area of semicircle} \\
 &\quad - \text{Area of triangle } ABC \\
 &= \sqrt{4} \left[\frac{2x^{3/2}}{3} \right]_0^{(\sqrt{3}+1)^2} - \frac{\pi}{2} - \frac{1}{2} ((\sqrt{3}+1)^2 - 2) 2(\sqrt{3}+1) \\
 &= \frac{4(\sqrt{3}+1)^3}{3} - \frac{\pi}{2} - 2(\sqrt{3}+1)^2
 \end{aligned}$$

ILLUSTRATION 9.24

Sketch the region bounded by the curves $y = x^2$ and $y = \frac{2}{1+x^2}$. Find the area.

Sol. The given curves are $y = x^2$ (1)

and $y = \frac{2}{1+x^2}$ (2)

Solving (1) and (2), we have

$$x^2 = \frac{2}{1+x^2}$$

or $x^4 + x^2 - 2 = 0$

or $(x^2 - 1)(x^2 + 2) = 0$

or $x = \pm 1$

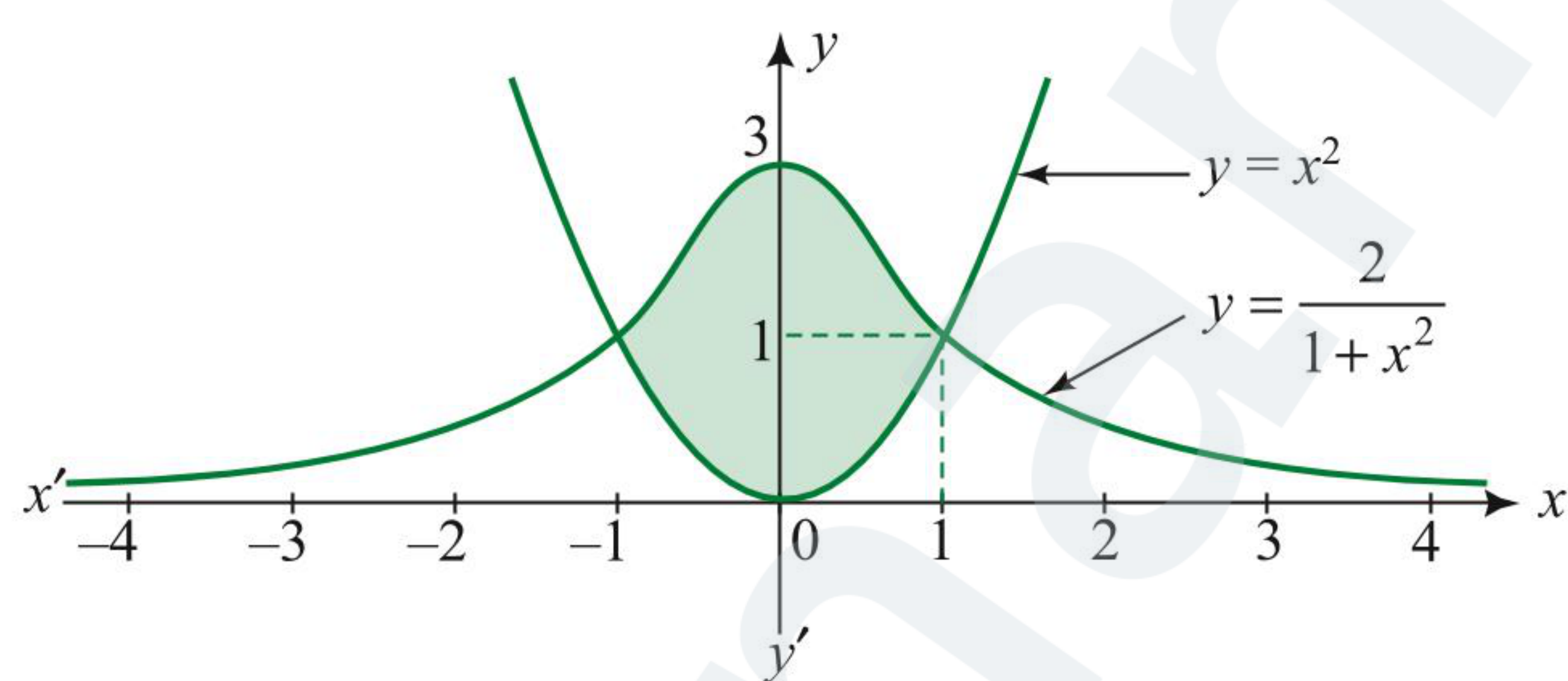
Also, $y = \frac{2}{1+x^2}$ is an even function.

Hence, its graph is symmetrical about y -axis.

At $x = 0$, $y = 2$, by increasing the values of x , y decreases and when $x \rightarrow \infty$, $y \rightarrow 0$.

Therefore, $y = 0$ is an asymptote of the given curve.

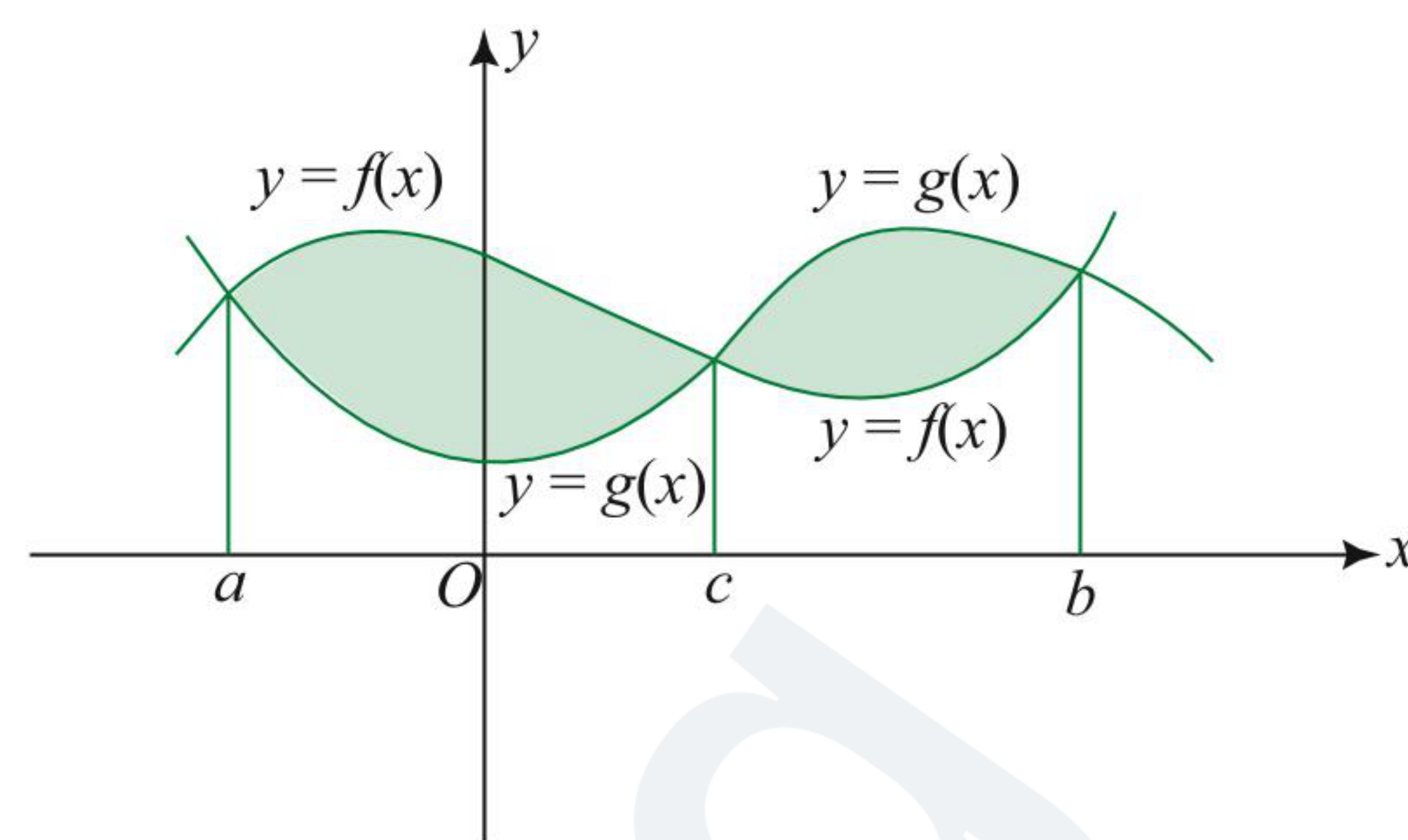
Thus, the graphs of the two curves are as follows:



$$\begin{aligned}
 \text{Required area} &= 2 \int_0^1 \left(\frac{2}{1+x^2} - x^2 \right) dx \\
 &= \left(4 \tan^{-1} x - \frac{2x^3}{3} \right)_0^1 \\
 &= \pi - \frac{2}{3} \text{ sq. units.}
 \end{aligned}$$

AREA BOUNDED BY TWO CURVES WHEN CURVES INTERSECT AT MORE THAN TWO POINTS

Consider the following graphs of the functions $y = f(x)$ and $y = g(x)$ which intersect at three points where $x = a$, $x = c$ and $x = b$.



Here $x = a$, $x = b$ and $x = c$, roots of the equation $f(x) = g(x)$.

For $x \in (a, c)$, $f(x) > g(x)$ and for $x \in (c, b)$, $g(x) > f(x)$.

Area bounded by curves,

$$\begin{aligned}
 A &= \int_a^b |f(x) - g(x)| \, dx \\
 &= \int_a^c (f(x) - g(x)) \, dx + \int_c^b (g(x) - f(x)) \, dx
 \end{aligned}$$

ILLUSTRATION 9.25

Find the area bounded by the curves $y = x^3 - x$ and $y = x^2 + x$.

Sol. $y = x^3 - x = x(x-1)(x+1)$ is a cubic polynomial

function intersecting the x -axis at $(-1, 0)$, $(0, 0)$, $(1, 0)$.

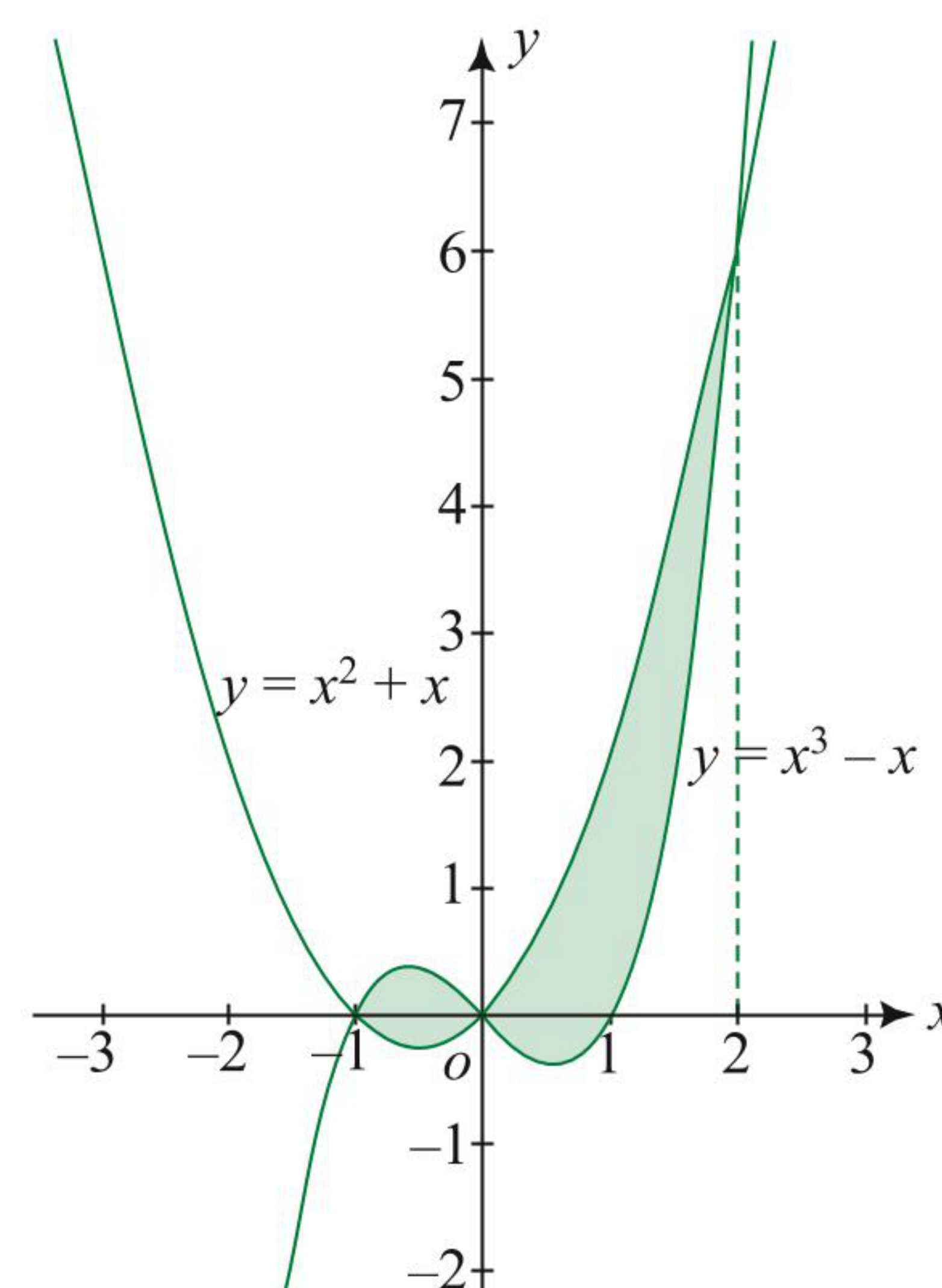
$y = x^2 + x = x(x+1)$ is a quadratic function which is concave upward and intersect x -axis at $(-1, 0)$, $(0, 0)$.

Solving curves, we get

$$x^3 - x = x^2 + x$$

$$\therefore x = -1, 0, 2$$

The graphs of functions are as shown in the figure.



From the figure,

$$\begin{aligned}
 \text{Required area} &= \int_{-1}^0 ((x^3 - x) - (x^2 + x)) \, dx + \int_0^2 (x^2 + x - (x^3 - x)) \, dx \\
 &= \int_{-1}^0 (x^3 - x^2 - 2x) \, dx + \int_0^2 (x^2 + 2x - x^3) \, dx
 \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 + \left[\frac{x^3}{3} + x^2 - \frac{x^4}{4} \right]_0^2 \\
&= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1 \right) \right] + \left[\frac{8}{3} + 4 - 4 \right] \\
&= \frac{37}{12} \text{ sq. units}
\end{aligned}$$

ILLUSTRATION 9.26

Find the area bounded by $y = -x^3 + x^2 + 16x$ and $y = 4x$

Sol. Solving given curve, we have

$$-x^3 + x^2 + 16x = 4x$$

$$\therefore x = 0 \text{ or } x^2 - x - 12 = 0$$

$$\Rightarrow x = 0 \text{ or } (x-4)(x+3) = 0$$

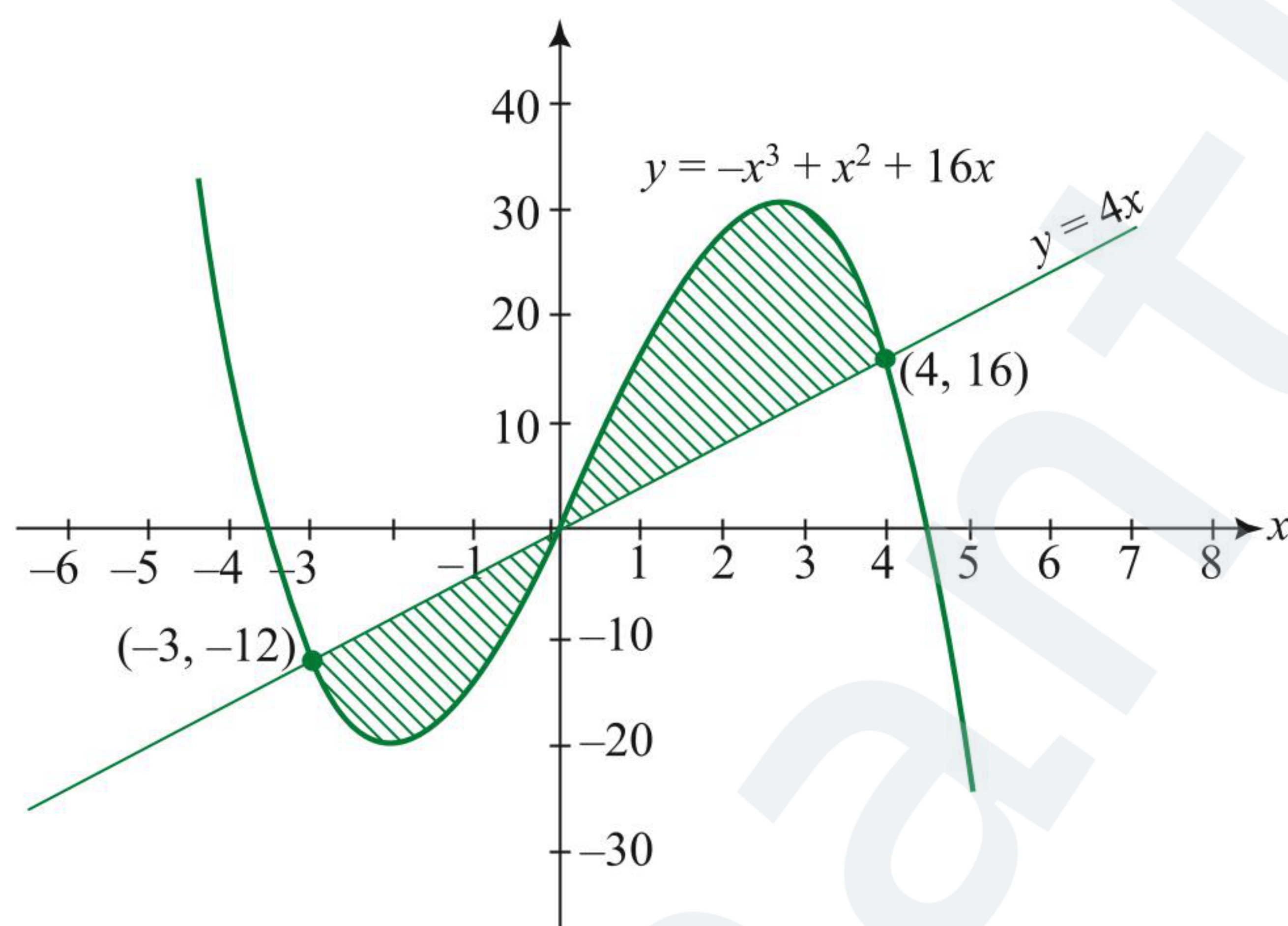
$$\Rightarrow x = 0 \text{ or } x = -3, 4$$

$$\begin{aligned} \text{Now } y &= -x^3 + x^2 + 16x \\ &= -x(x^2 - x - 16) \end{aligned}$$

$$\text{If } y = 0 \text{ then } x = 0 \text{ or } x^2 - x - 16 = 0$$

$$\therefore x = 0 \text{ or } x = \frac{1 \pm \sqrt{65}}{2}$$

From the above information we having following graphs.



From the figure area of the shaded region,

$$\begin{aligned}
A &= \int_{-3}^0 [4x - (-x^3 + x^2 + 16x)] dx + \int_0^4 [(-x^3 + x^2 + 16x) - 4x] dx \\
&= \int_{-3}^0 (x^3 - x^2 - 12x) dx + \int_0^4 (-x^3 + x^2 + 12x) dx \\
&= \left[\frac{x^4}{4} - \frac{x^3}{3} - 6x^2 \right]_{-3}^0 + \left[-\frac{x^4}{4} + \frac{x^3}{3} + 6x^2 \right]_0^4 \\
&= \left[-\frac{81}{4} - 9 + 54 \right] + \left[-64 + \frac{64}{3} + 96 \right] \\
&= \frac{937}{12} \text{ sq. units.}
\end{aligned}$$

ILLUSTRATION 9.27

Find the area of the region enclosed by $y = 5x - x^2$ and $y = x$ on interval $[-1, 5]$

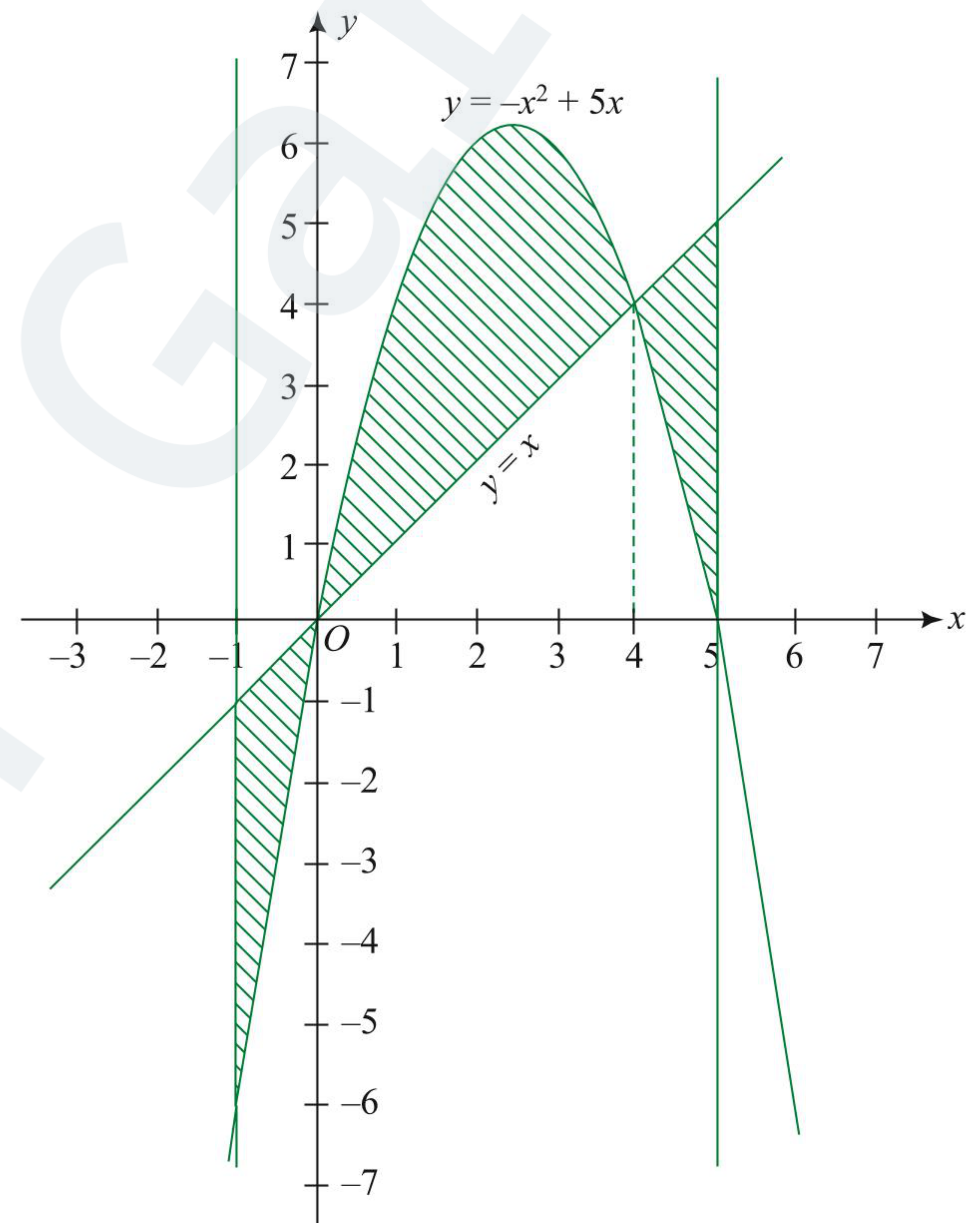
Sol. Solving the given curves we get

$$5x - x^2 = x$$

$$\therefore x^2 - 4x = 0$$

$$\Rightarrow x = 0, 4$$

$y = x(5 - x)$ is parabola which is concave downward and intersecting x -axis at $x = 0$ and $x = 5$.



$$\text{Required area, } = \int_{-1}^5 |x - (5x - x^2)| dx$$

$$\begin{aligned}
&= \int_{-1}^5 |x^2 - 4x| dx \\
&= \int_{-1}^0 (x^2 - 4x) dx + \int_0^4 (4x - x^2) dx + \int_4^5 (x^2 - 4x) dx \\
&= \left[\frac{x^3}{3} - 2x^2 \right]_{-1}^0 + \left[2x^2 - \frac{x^3}{3} \right]_0^4 + \left[\frac{x^3}{3} - 2x^2 \right]_4^5 \\
&= -\left[\frac{-1}{3} - 2 \right] + \left[32 - \frac{64}{3} \right] + \left[\frac{125}{3} - 50 \right] - \left[\frac{64}{3} - 32 \right] \\
&= 16 - \frac{2}{3} \\
&= \frac{46}{3} \text{ sq. units}
\end{aligned}$$

CONCEPT APPLICATION EXERCISE 9.2

- Find the area lying in the first quadrant and bounded by the curve $y = x^3$ and the line $y = 4x$.
- Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.
- Find the area enclosed by the figure described by the equation $x^4 + 1 = 2x^2 + y^2$.
- In what ratio does the x -axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?
- Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.
- Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 3$.
- Find the area of the region bounded by the limits $x = 0$, $x = \frac{\pi}{2}$, and $f(x) = \sin x$, $g(x) = \cos x$.
- Find the area bounded by $y = \tan^{-1} x$, $y = \cot^{-1} x$, and y -axis in the first quadrant.
- Find the area bounded by $y = \log_e x$, $y = -\log_e x$, $y = \log_e(-x)$, and $y = -\log_e(-x)$.
- Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$.
- Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area.
- Sketch the curves and identify the region bounded by $x = \frac{1}{2}$, $x = 2$, $y = \ln x$, and $y = 2^x$. Find the area of this region.
- Find the area bounded by $y = x^2$ and $y = x^{1/3}$ for $x \in [-1, 1]$.
- Find the smallest area bounded by the curves $y = x - \sin x$, $y = x + \cos x$.

ANSWERS

- 4 sq. units
- $\frac{9}{8}$ sq. units
- $\frac{8}{3}$ sq. units
- 121 : 4
- $\frac{4}{3}(8\pi - \sqrt{3})$ sq. units
- $\frac{21}{2}$ sq. units
- $2(\sqrt{2} - 1)$ sq. units
- $\log \sqrt{2}$ sq. units
- 4 sq. units
- $2\left(\frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}\right)$ sq. units
- $\frac{5\pi - 2}{4}$ sq. units
- $\left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right)$ sq. units
- $\frac{3}{2}$ sq. units
- $2\sqrt{2}$ sq. units

AREA BOUNDED BY MISCELLANEOUS CURVES

CONCEPT OF VARIABLE AREA

Till now we have been finding the area bounded by curves using integration. In the reverse situation if area bounded by curves in which one of the curves is unknown is given with one of the limits variable, we can find the function using differentiation.

ILLUSTRATION 9.28

If the area enclosed by curve $y = f(x)$ and $y = x^2 + 2$ between the abscissa $x = 2$ and $x = \alpha$, $\alpha > 2$, is $(\alpha^3 - 4\alpha^2 + 8)$ sq. units then find function $f(x)$. It is known that curve $y = f(x)$ lies below the parabola $y = x^2 + 2$.

Sol. According to the question,

$$\alpha^3 - 4\alpha^2 + 8 = \int_2^\alpha (x^2 + 2 - f(x)) dx$$

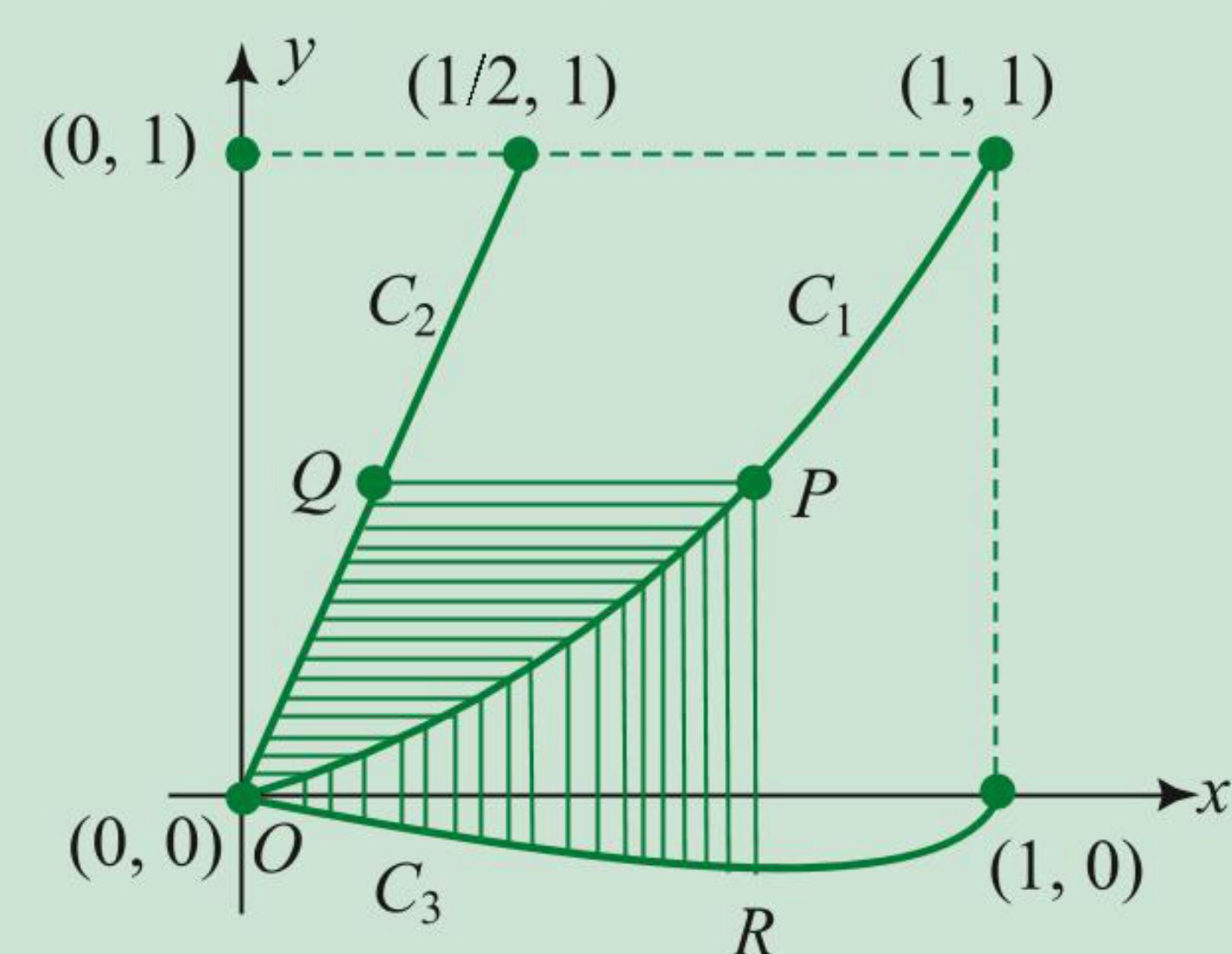
Differentiating w.r.t. α on both sides, we get

$$3\alpha^2 - 8\alpha = \alpha^2 + 2 - f(\alpha)$$

$$\therefore f(x) = -2x^2 + 8x + 2$$

ILLUSTRATION 9.29

Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, respectively, where $0 \leq x \leq 1$. Let C_3 be the graph of a function $y = f(x)$, where $0 \leq x \leq 1$, $f(0) = 0$. For a point P on C_1 , let the lines through P , parallel to the axes, meet C_2 and C_3 at Q and R , respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$.



Sol. Let P be on C_1 , $y = x^2$ be (t, t^2)

$\therefore$ y co-ordinate of Q is also t^2

Now, Q on $y = 2x$ where $y = t^2$

$$\therefore x = t^2/2$$

$$\therefore Q\left(\frac{t^2}{2}, t^2\right)$$

For point R , $x = t$ and it is on $y = f(x)$

$$\therefore R(t, f(t))$$

Given that,

$$\text{Area } OPQ = \text{Area } OPR$$

$$\Rightarrow \int_0^{t^2} \left(\sqrt{y} - \frac{y}{2}\right) dy = \int_0^t (x^2 - f(x)) dx$$

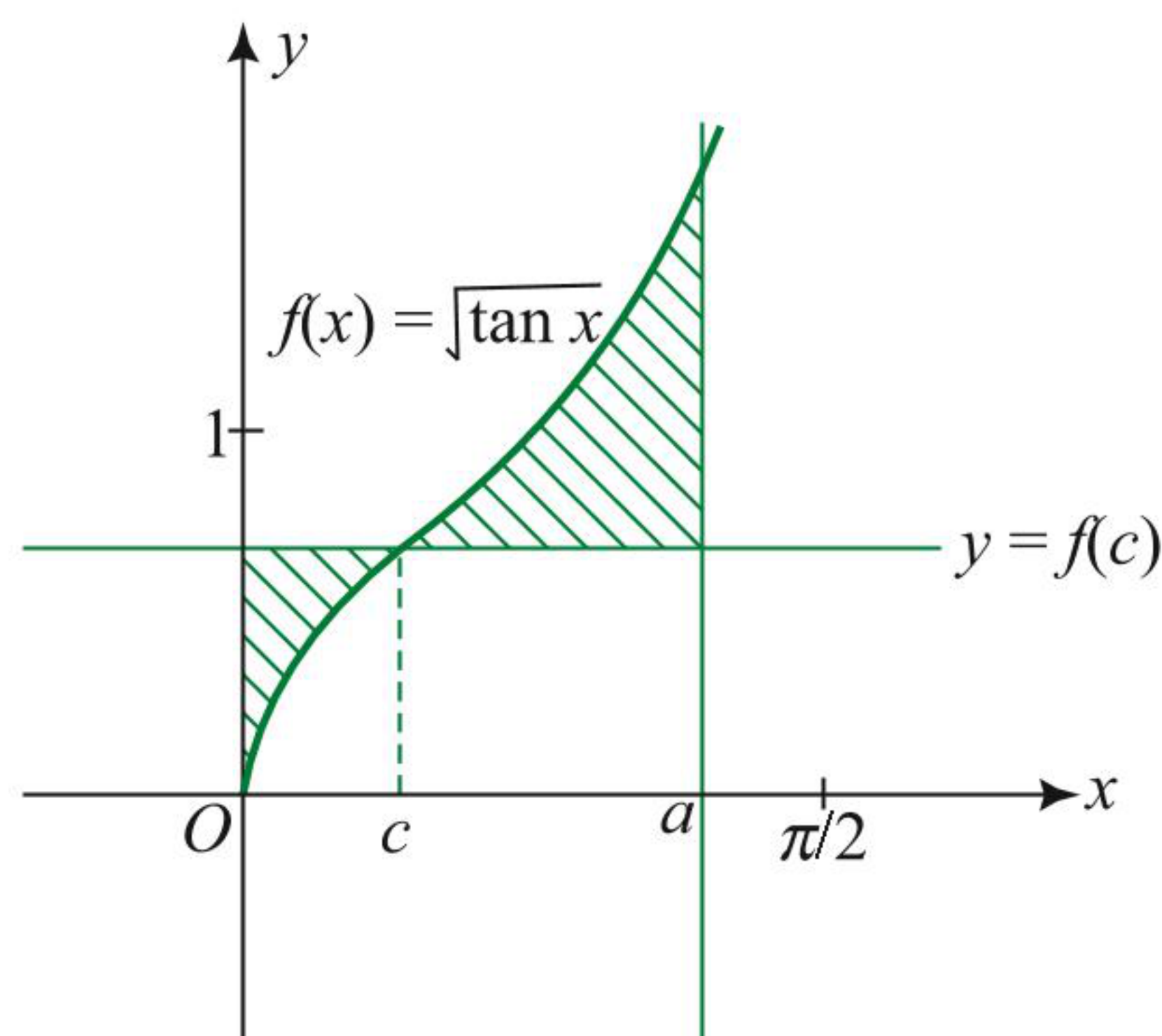
Differentiating both sides w.r.t. t , we get

$$\begin{aligned} \left(\sqrt{t^2} - \frac{t^2}{2} \right) (2t) &= t^2 - f(t) \\ \Rightarrow f(t) &= t^3 - t^2 \\ \Rightarrow f(x) &= x^3 - x^2 \end{aligned}$$

ILLUSTRATION 9.30

If the area bounded by $f(x) = \sqrt{\tan x}$, $y = f(c)$, $x = 0$ and $x = a$, $0 < c < a < \frac{\pi}{2}$ is minimum then find the value of c .

Sol. Graph of $f(x) = \sqrt{\tan x}$ is as shown in the following figure.



Required area is

$$\begin{aligned} A &= \int_0^c (f(c) - f(x)) dx + \int_c^a (f(x) - f(c)) dx \\ &= \int_0^c \sqrt{\tan c} dx - \int_0^c \sqrt{\tan x} dx + \int_c^a \sqrt{\tan x} dx - \int_c^a \sqrt{\tan c} dx \\ &= (2c - a)\sqrt{\tan c} - \int_0^c \sqrt{\tan x} dx + \int_c^a \sqrt{\tan x} dx \end{aligned}$$

$$\begin{aligned} \therefore \frac{dA}{dc} &= 2\sqrt{\tan c} + (2c - a) \frac{\sec^2 c}{2\sqrt{\tan c}} - \sqrt{\tan c} - \sqrt{\tan c} \\ &= (2c - a) \frac{\sec^2 c}{2\sqrt{\tan c}} \end{aligned}$$

$$\frac{dA}{dc} = 0 \Rightarrow c = \frac{a}{2}$$

At $c = \frac{a}{2}$, $\frac{dA}{dc}$ changes sign from negative to positive.

Hence A is minimum when $c = \frac{a}{2}$

AREA BOUNDED BY INEQUALITIES

ILLUSTRATION 9.31

Find the area of the region R which is enclosed by the curves $y \geq \sqrt{1-x^2}$ and $\max\{|x|, |y|\} \leq 4$.

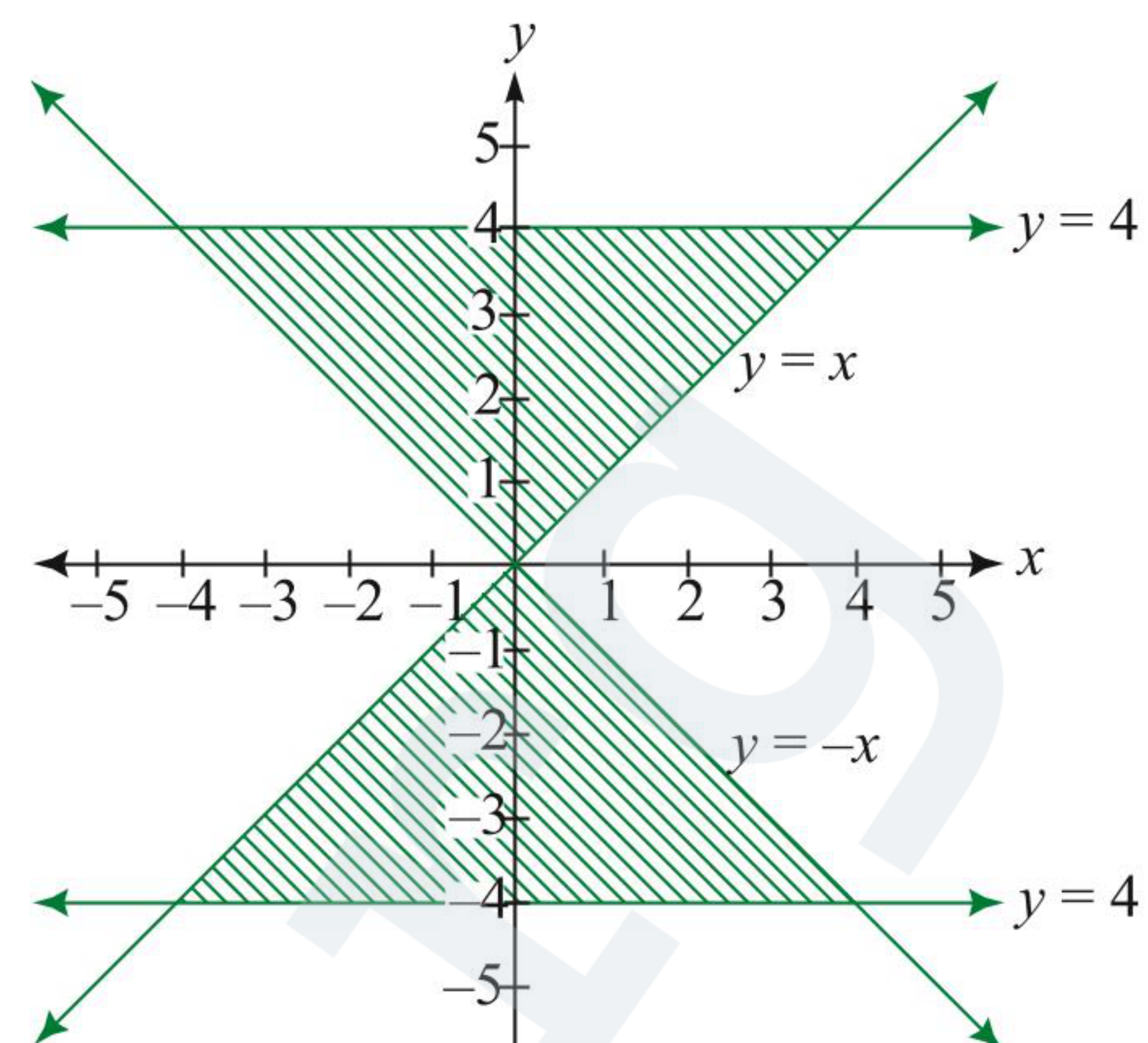
Sol. For $y \geq \sqrt{1-x^2}$ (1)

points lie outside circle $x^2 + y^2 = 1$ for $y \geq 0$ and $-1 \leq x \leq 1$.

For $\max\{|x|, |y|\} \leq 4$, we have

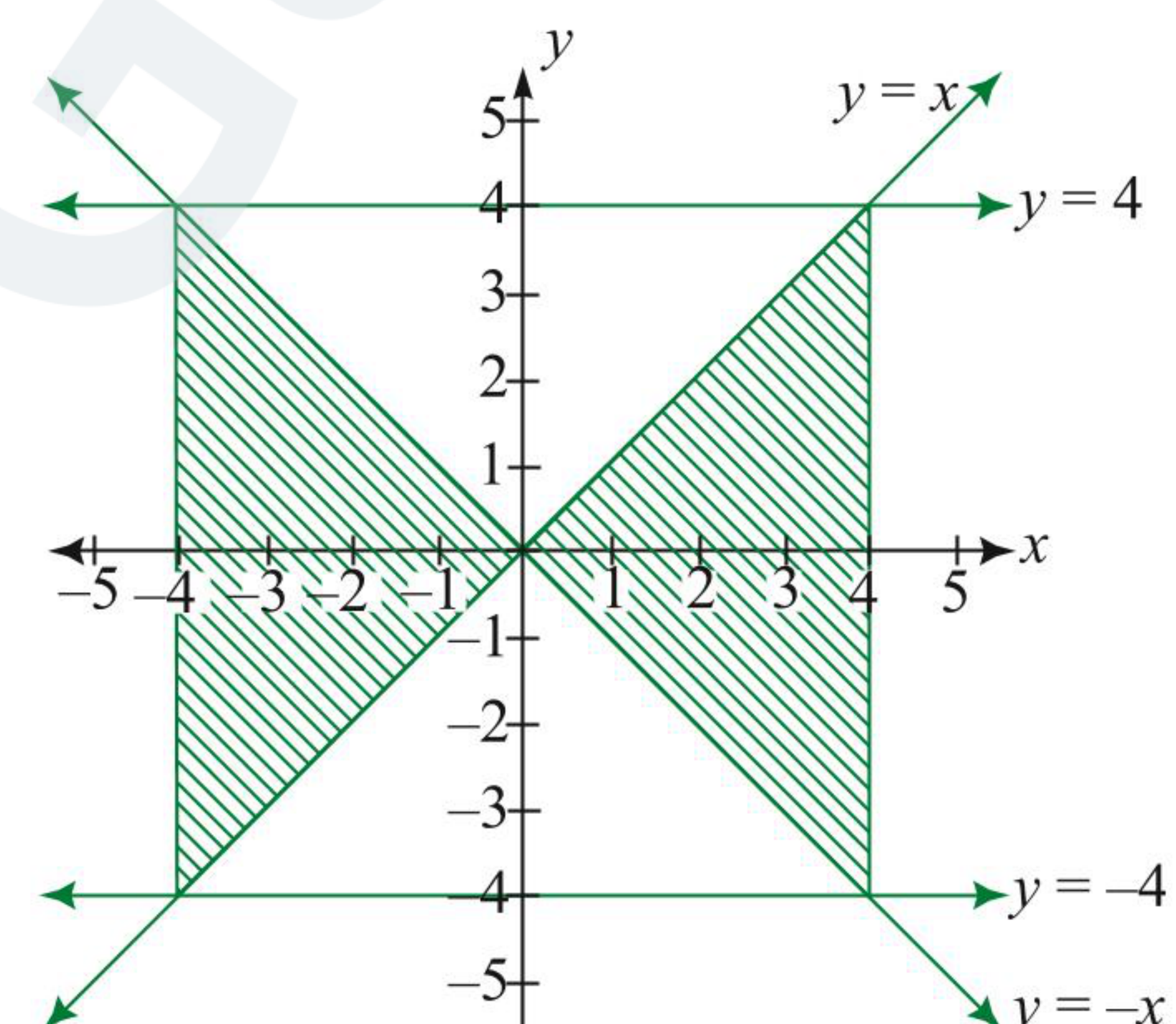
$|y| \leq 4$, when $|x| < |y|$ (2)

The points satisfying above inequalities from region as shown in the figure.

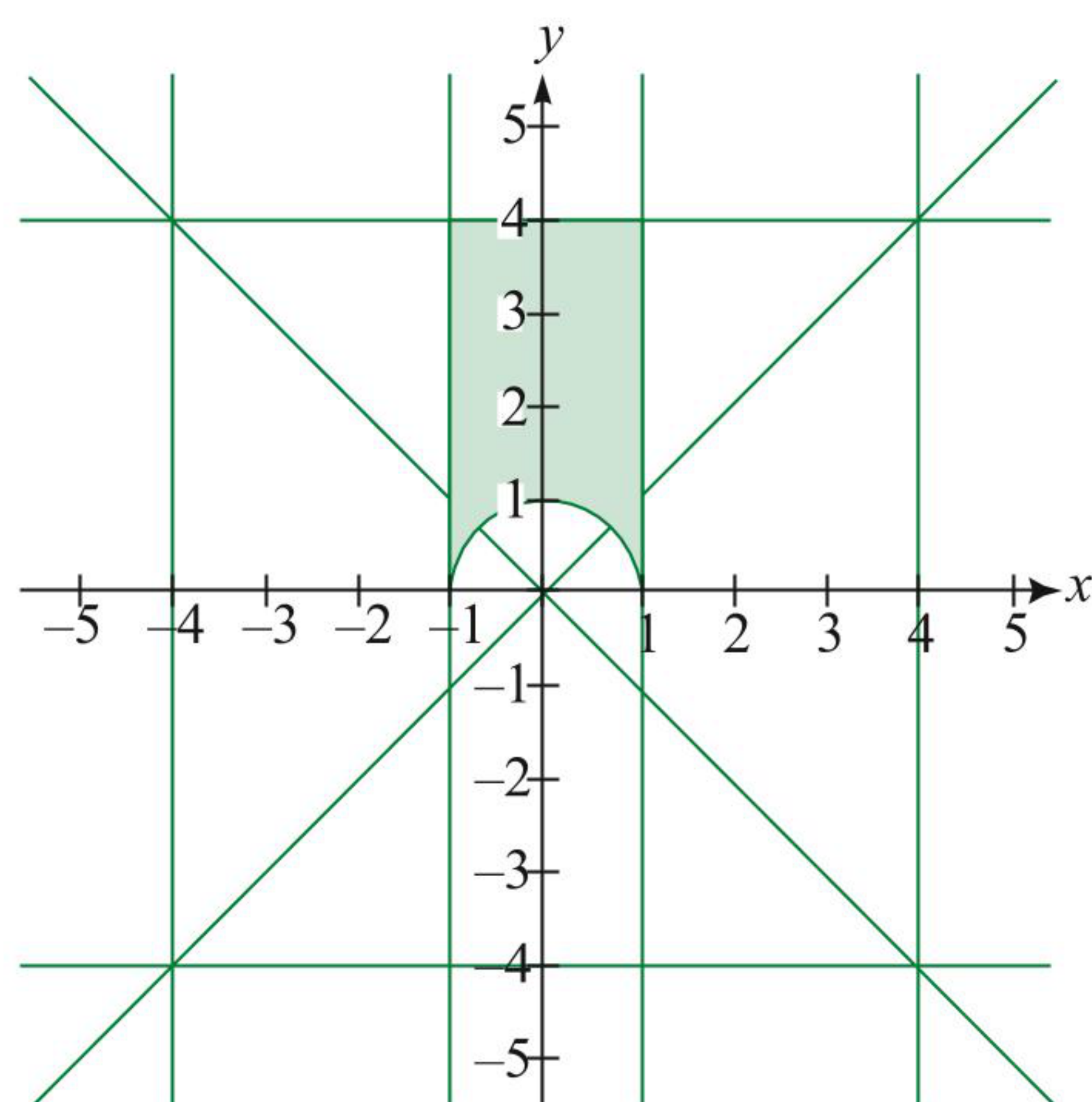


$|x| \leq 4$, when $|y| < |x|$ (3)

The points satisfying above inequalities from region as shown in the figure.



The region $(1) \cap ((2) \cup (3))$ is as shown in the figure.



From the figure, required area is

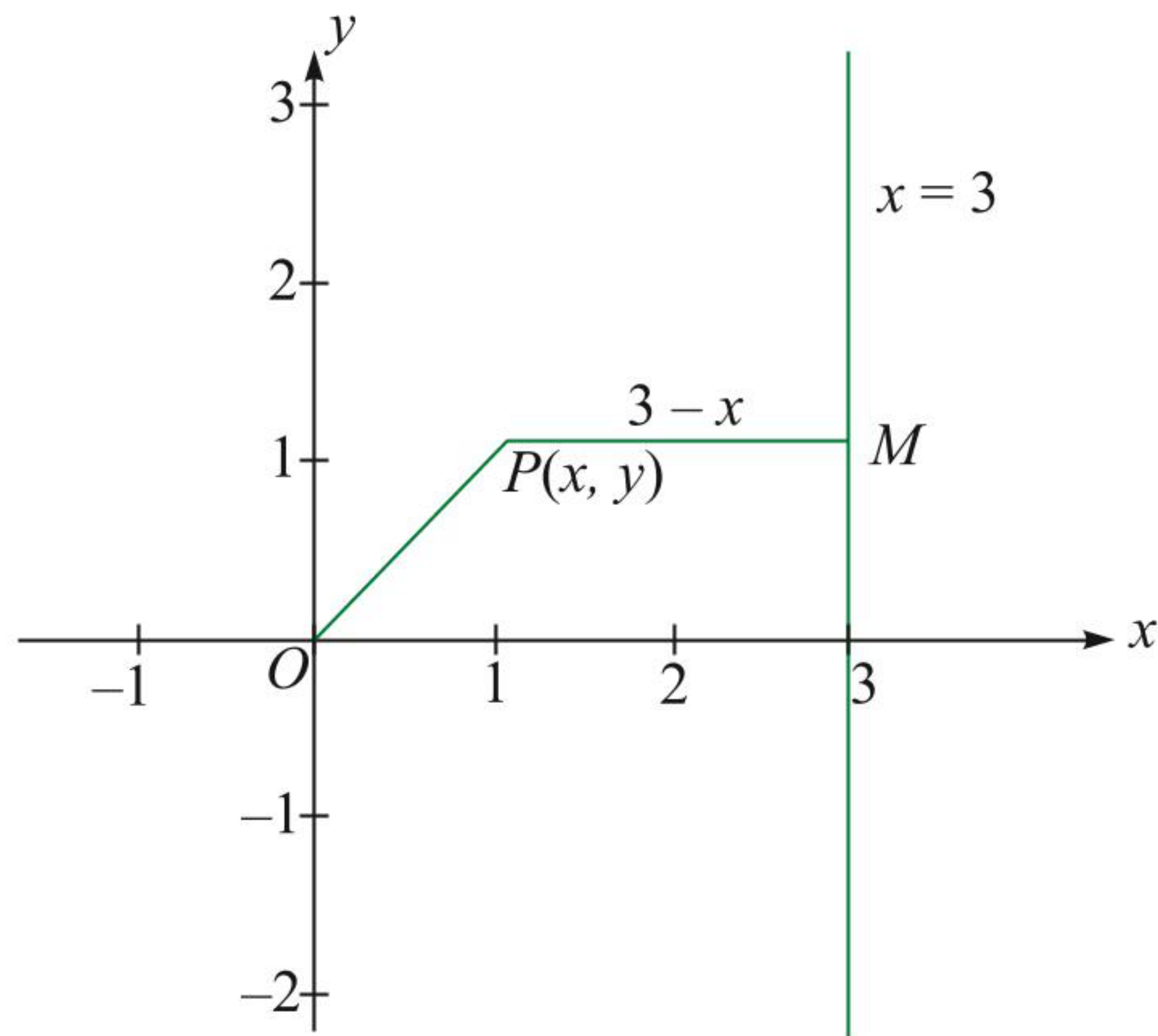
= Area of rectangle – Area of semicircle having radius 1

$$= 8 - \frac{\pi}{2}$$

ILLUSTRATION 9.32

Plot the region in the first quadrant in which points are nearer to the origin than to the line $x = 3$.

Sol. Point $P(x, y)$ lies in the first quadrant. Therefore, $x, y > 0$



Also point $P(x, y)$ is nearer to the origin than to the line $x = 3$

$$\text{Now } OP = \sqrt{x^2 + y^2}$$

Distance of P from line $x - 3 = 0$ is $PM = 3 - x$

According to the question, $OP < PM$

$$\Rightarrow \sqrt{x^2 + y^2} < (3 - x)$$

$$\text{or } x^2 + y^2 < x^2 - 6x + 9$$

$$\text{or } y^2 < -6x + 9$$

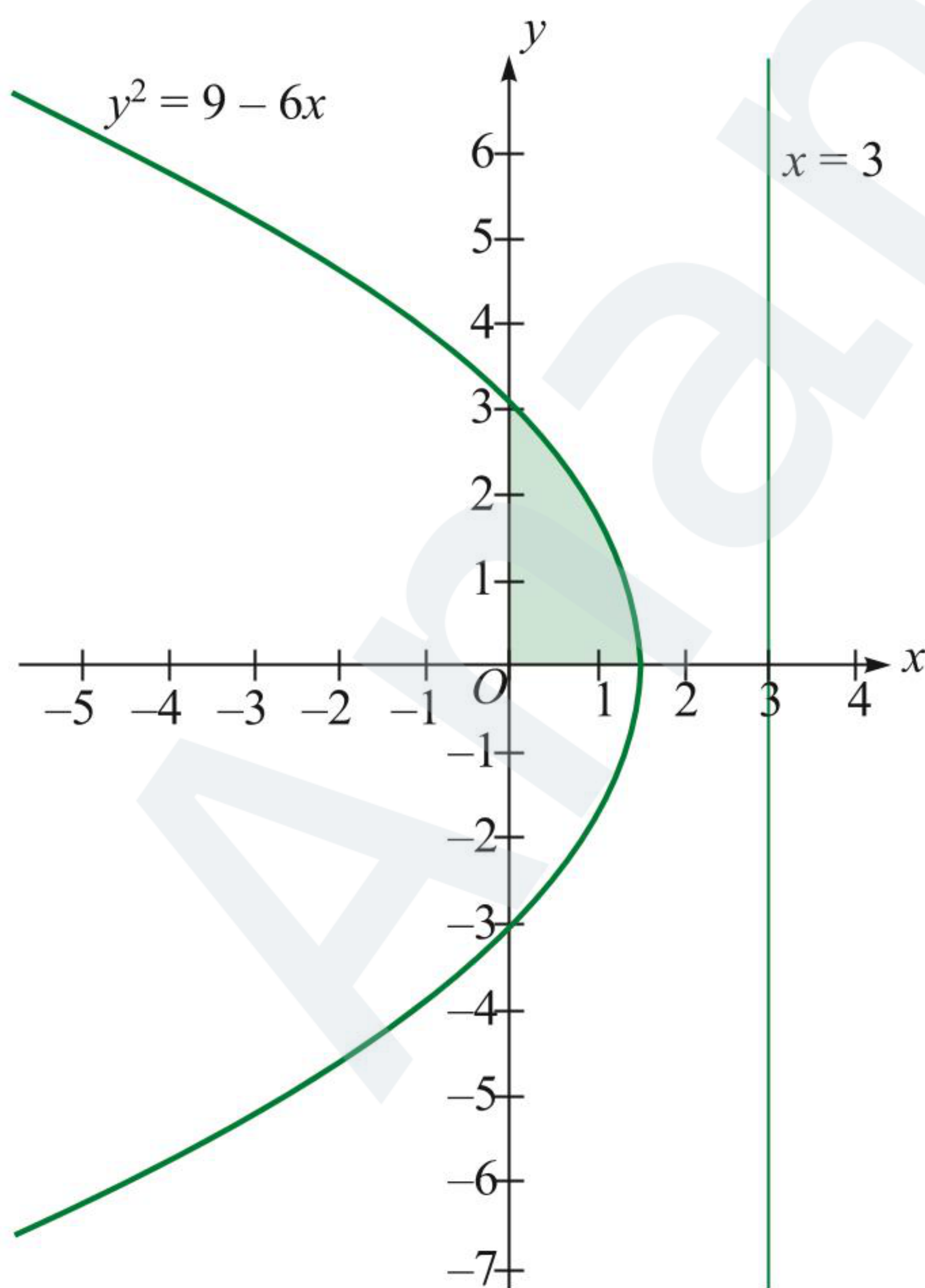
$$\text{or } y^2 < -6\left(x - \frac{3}{2}\right)$$

Points satisfying above inequality lie inside parabola $y^2 = 9 - 6x$ in the first quadrant.

Parabola $y^2 = -6\left(x - \frac{3}{2}\right)$ is concave to the left, having axis as

x -axis and vertex at $\left(\frac{3}{2}, 0\right)$ and directrix $x = 3$

The required region as shown in the figure.

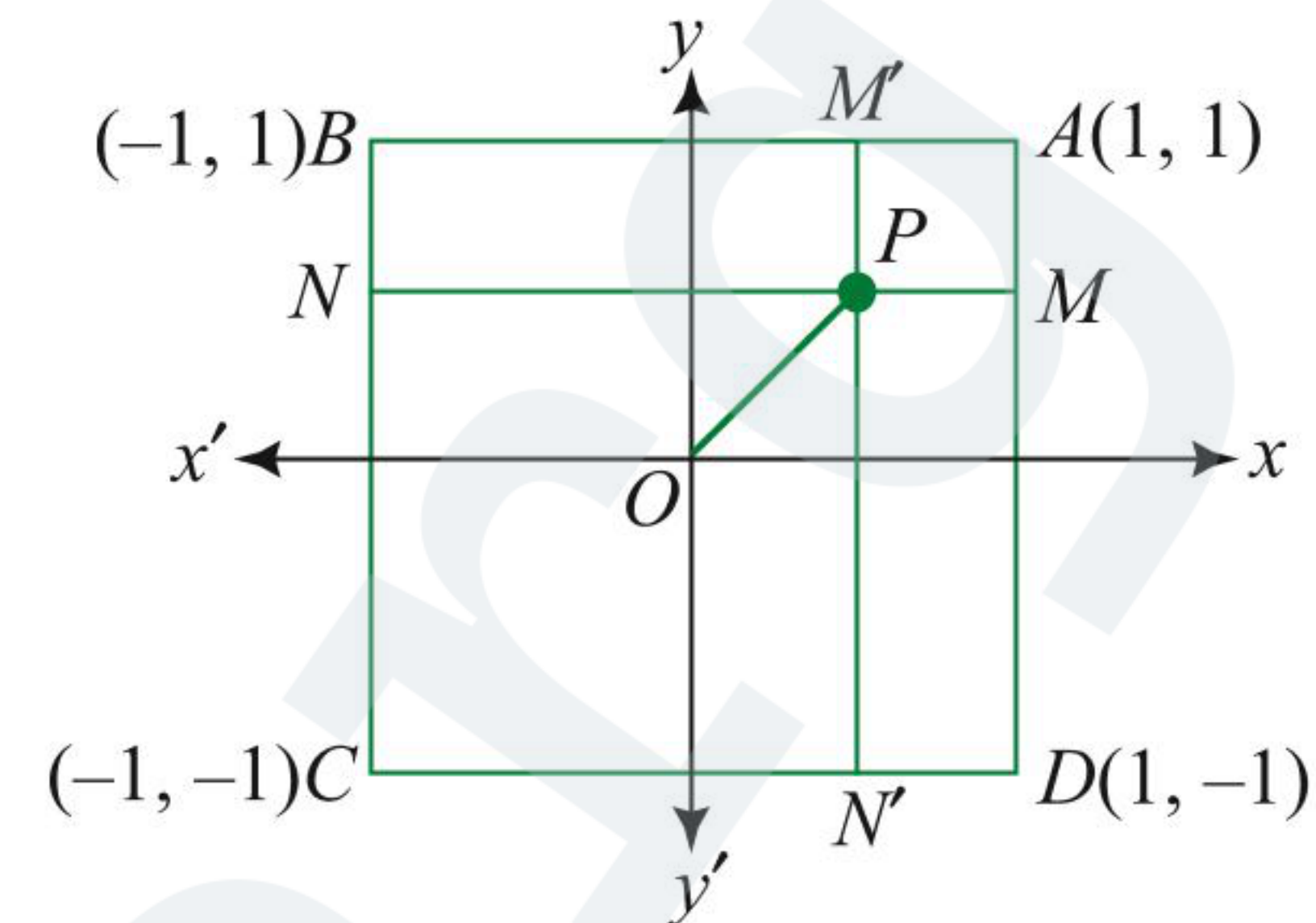


$$\begin{aligned} \text{Area of the region} &= \int_0^3 \frac{9 - y^2}{6} dy = \frac{1}{6} \left(9y - \frac{y^3}{3} \right)_0^3 \\ &= \frac{1}{6} (27 - 9) = 3 \text{ sq. units} \end{aligned}$$

ILLUSTRATION 9.33

Consider a square with vertices at $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$. Let S be the region consisting of all points inside the square which are nearer to the origin than to any of the edges. Sketch the region S and find its area.

Sol.



Let us consider any point (x, y) inside the square such that its distance from origin is less than its distance from any of the edges.

Consider edge AD . Therefore,

$$OP < PM$$

$$\Rightarrow \sqrt{x^2 + y^2} < 1 - x \text{ or } y^2 < -2\left(x - \frac{1}{2}\right) \quad (1)$$

Above represents all points within the parabola $y^2 = 1 - 2x$.

If we consider the edge BC , then $OP < PN$ will imply

$$y^2 < 2\left(x + \frac{1}{2}\right) \quad (2)$$

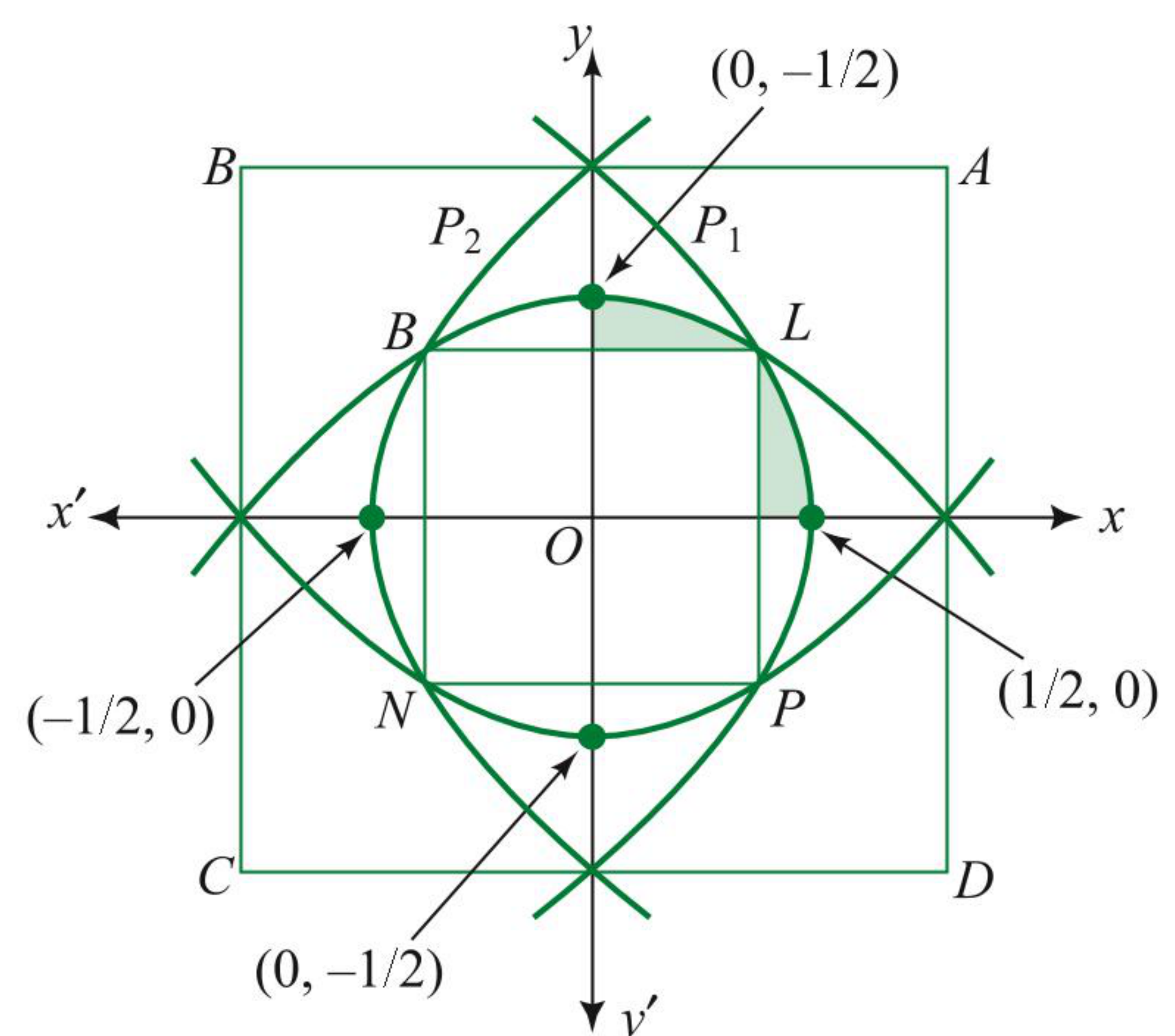
Similarly, if we consider the edges AB and CD , we will have

$$x^2 < -2\left(y - \frac{1}{2}\right) \quad (3)$$

$$x^2 < 2\left(y + \frac{1}{2}\right) \quad (4)$$

Hence, S consists of the region bounded by four parabolas meeting the axes at $\left(\pm \frac{1}{2}, 0\right)$ and $\left(0, \pm \frac{1}{2}\right)$.

The point L is intersection of P_1 and P_3 given by (1) and (3).



Solving, we get

$$y^2 - x^2 = -2(x - y) = 2(y - x)$$

$$\text{or } y - x = 0$$

$$\text{or } y = x$$

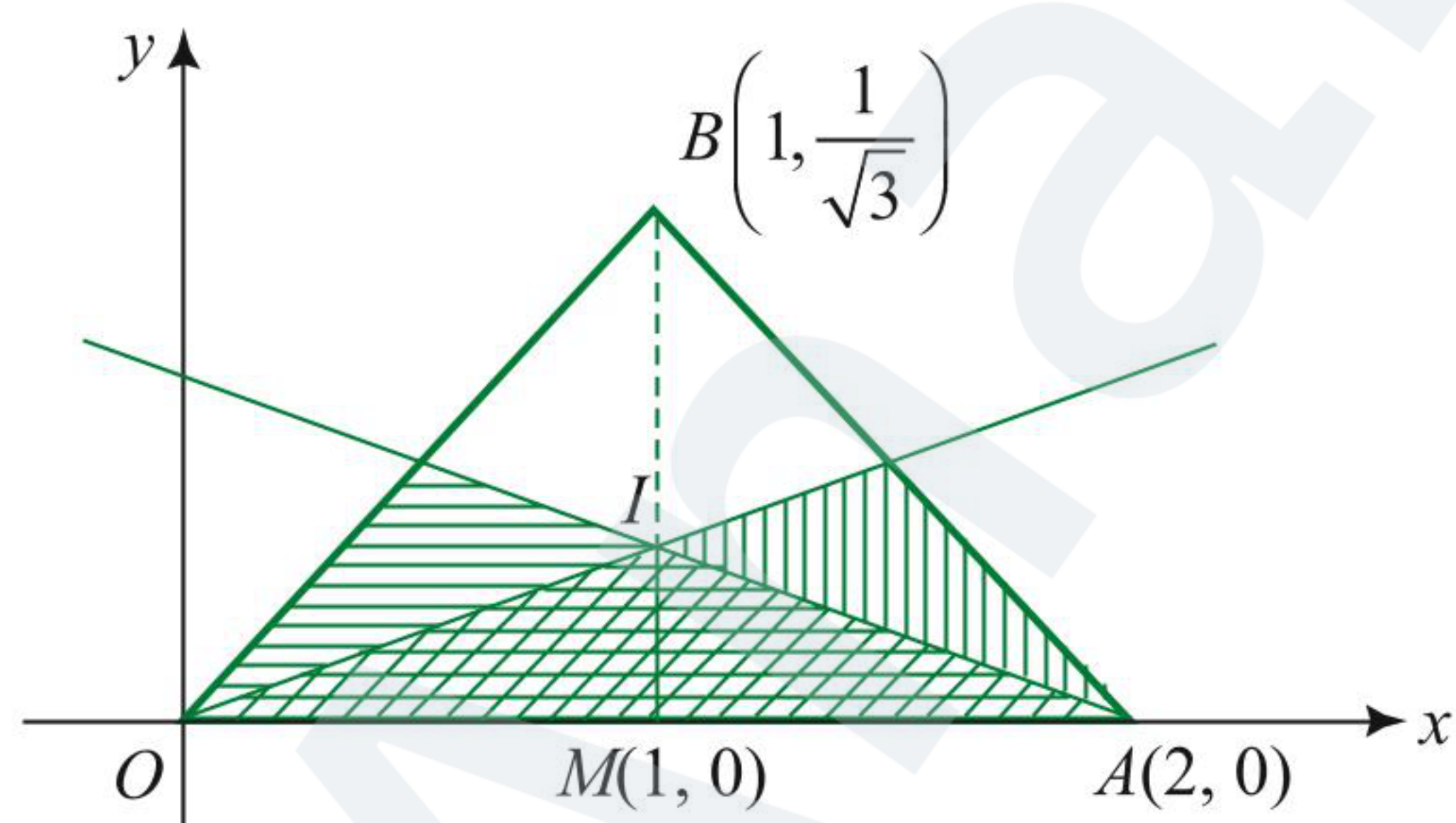
$$\begin{aligned} \Rightarrow x^2 + 2x - 1 &= 0 \\ \text{or } (x+1)^2 &= 2 \\ \text{or } x &= \sqrt{2} - 1 \text{ as } x \text{ is +ve} \\ \therefore L &\text{ is } (\sqrt{2} - 1, \sqrt{2} - 1). \end{aligned}$$

$$\begin{aligned} \therefore \text{Total area} &= 4 \left[\text{Square of side } (\sqrt{2} - 1) + 2 \int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x} dx \right] \\ &= 4 \left\{ (\sqrt{2} - 1)^2 + 2 \int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x} dx \right\} \\ &= 4 \left[3 - 2\sqrt{2} - \frac{2}{3} \left\{ (1-2x)^{3/2} \right\}_{\sqrt{2}-1}^{1/2} \right] \\ &= 4 \left[3 - 2\sqrt{2} - \frac{2}{3} \left\{ 0 - (1 - 2\sqrt{2} + 2)^{3/2} \right\} \right] \\ &= 4 \left[3 - 2\sqrt{2} + \frac{2}{3} (3 - 2\sqrt{2})^{3/2} \right] \\ &= 4(3 - 2\sqrt{2}) \left[1 + \frac{2}{3} \sqrt{3 - 2\sqrt{2}} \right] \\ &= 4(3 - 2\sqrt{2}) \left[1 + \frac{2}{3} (\sqrt{2} - 1) \right] \\ &= \frac{4}{3} (3 - 2\sqrt{2}) (1 + 2\sqrt{2}) = \frac{4}{3} [(4\sqrt{2} - 5)] \\ &= \frac{16\sqrt{2} - 20}{3} \text{ sq. units.} \end{aligned}$$

ILLUSTRATION 9.34

Let $O(0, 0)$, $A(2, 0)$, and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle.

Let R be the region consisting of all those points P inside $\triangle OAB$ which satisfy $d(P, OA) \leq \min [d(P, OB), d(P, AB)]$, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

Sol.

$$d(P, OA) \leq \min [d(P, OB), d(P, AB)]$$

$$\Rightarrow d(P, OA) \leq d(P, OB) \text{ and } d(P, OA) \leq d(P, AB)$$

When $d(P, OA) = d(P, OB)$, P is equidistant from OA and OB , or P lies on the angular bisector of lines OA and OB .

Hence, when $d(P, OA) \leq d(P, OB)$, point P is nearer to OA than to OB , i.e., lies on or below the bisector of OA and OB .

Similarly, when $d(P, OA) \leq d(P, AB)$, P is nearer to OA than to OB , i.e., lies on or below the bisector of OA and AB .

$$\therefore \text{Required area} = \text{Area of } \triangle OIA.$$

$$\text{Now, } \tan \angle BOA = \frac{1/\sqrt{3}}{1} = \frac{1}{\sqrt{3}}$$

$$\text{or } \angle BOA = 30^\circ \Rightarrow \angle IOA = 15^\circ$$

$$\Rightarrow IM = \tan 15^\circ = 2 - \sqrt{3}.$$

$$\begin{aligned} \text{Hence, Area of } \triangle OIA &= \frac{1}{2} OA \times IM = \frac{1}{2} \times 2 \times (2 - \sqrt{3}) \\ &= 2 - \sqrt{3} \text{ sq. units} \end{aligned}$$

AREA BOUNDED BY INVERSE OF GIVEN FUNCTION WITHOUT GETTING INVERSE FUNCTION

In some problems, we have to find the bounded area which involves inverse function of the given function and other curves. But what if the inverse function cannot be determined? Well, in this case still we can find the area using substitution method using and integration by parts.

ILLUSTRATION 9.35

Find the area enclosed by $y = g(x)$, x -axis, $x = 1$ and $x = 37$, where $g(x)$ is inverse of $f(x) = x^3 + 3x + 1$.

$$\text{Sol. Required area, } A = \int_1^{37} g(x) dx = \int_1^{37} f^{-1}(x) dx.$$

$$\text{Let } f^{-1}(x) = t \text{ or } x = f(t)$$

Using intelligent guessing, $f(3) = 37$ and $f(0) = 1$

$$\begin{aligned} \therefore A &= \int_0^3 t f'(t) dt = [t f(t)]_0^3 - \int_0^3 f(t) dt \\ &= 3 f(3) - \int_0^3 (t^3 + 3t + 1) dt \\ &= 111 - \frac{147}{4} = \frac{297}{4} \end{aligned}$$

Alternative method:

$$f(x) = x^3 + 3x + 1.$$

$$\therefore f'(x) = 3x^2 + 3 > 0, \forall x \in R.$$

$$\therefore f(x) \text{ is an increasing function.}$$

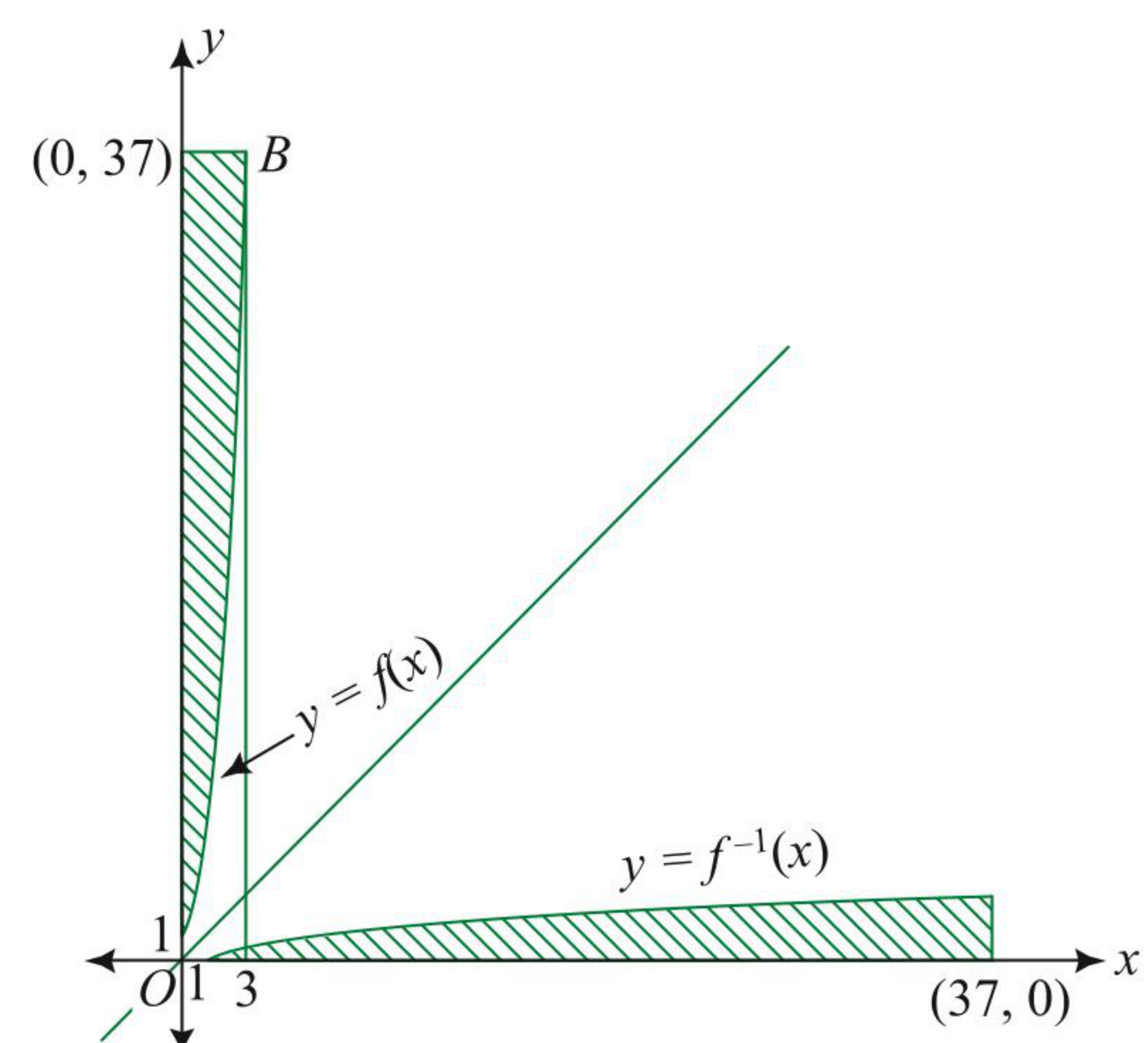
Also, $x^3 + 3x + 1 = x$ or $x^3 + 2x + 1 = 0$ has no positive root.

So, line $y = x$ never meet curve $y = f(x)$ for $x > 0$.

Graph of $y = f(x)$ and $y = f^{-1}(x)$ are as shown in the following figure.

When $y = 1$, $x^3 + 3x + 1 = 1$, $x = 0$.

When $y = 37$, $x^3 + 3x = 36$, $x = 3$



Area bounded by curves $y = f^{-1}(x)$, x -axis, $x = 1$ and $x = 37$ is same as area bounded by curves $y = f(x)$, y -axis, $y = 1$ and $y = 37$.

$$\begin{aligned}\therefore \text{ Required area} &= \int_0^3 (37 - x^3 - 3x - 1) dx \\ &= \left[36x - \frac{x^4}{4} - \frac{3x^2}{2} \right]_0^3 = \frac{297}{4} \text{ sq. units.}\end{aligned}$$

ILLUSTRATION 9.36

Find the area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$.

Sol. $y = x + \sin x$

$$\therefore \frac{dy}{dx} = 1 + \cos x \geq 0 \quad \forall x$$

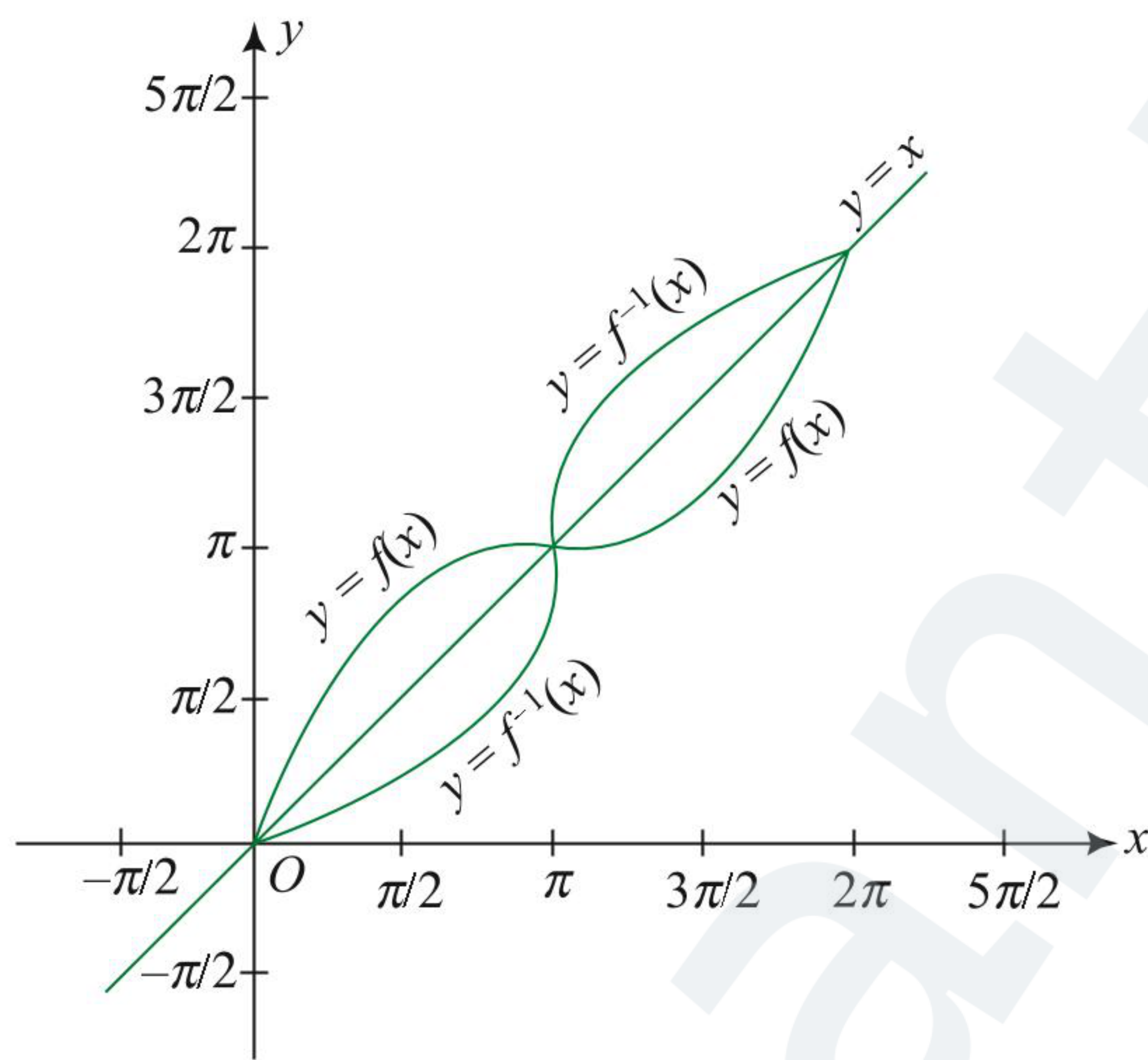
$$\text{Also } \frac{d^2y}{dx^2} = -\sin x = 0 \text{ when } x = n\pi, n \in \mathbb{Z}$$

Hence $x = n\pi$ are points of inflection, where curve changes its concavity.

Also, for $x \in (0, \pi)$, $\sin x > 0 \Rightarrow x + \sin x > x$

And for $x \in (\pi, 2\pi)$, $\sin x < 0 \Rightarrow x + \sin x < x$

Thus, we have following graphs of function $y = f(x)$ and its inverse function.



$$\begin{aligned}\text{Required area} &= 4 \int_0^\pi (f(x) - x) dx \\ &= 4 \int_0^\pi \sin x dx = 4 \times 2 = 8 \text{ sq. units.}\end{aligned}$$

CONCEPT APPLICATION EXERCISE 9.3

- Find a continuous function f , where $(x^4 - 4x^2) \leq f(x) \leq (2x^2 - x^3)$ such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, the y -axis, and the line $x = t$, where $(0 \leq t \leq 2)$ is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, y -axis, and line $x = t$ (where $0 \leq t \leq 2$).
- If the area bounded by the x -axis, the curve $y = f(x)$, ($f(x) > 0$) and the lines $x = 1$, $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$, then find $f(x)$.
- The area bounded by the graph of $y = f(x)$, $f(x) > 0$ on $[0, a]$ and x -axis is $\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$ then find the value of $f\left(\frac{\pi}{2}\right)$.
- A curve $y = f(x)$ is such that $f(x) \geq 0$ and $f(0) = 0$ and bounds a curvilinear triangle with the base $[0, x]$ whose area is proportional to $(n+1)^{\text{th}}$ power of $f(x)$. If $f(1) = 1$ then find $f(x)$.
- Find the area of curve enclosed by $|x + y| + |x - y| \leq 4$, $|x| \leq 1$, $y \geq \sqrt{x^2 - 2x + 1}$.
- Consider two regions
 R_1 : points P are nearer to $(1, 0)$ than to $x = -1$.
 R_2 : Points P are nearer to $(0, 0)$ than to $(8, 0)$
 Find the area of the region common to R_1 and R_2 .
- If $f: [-1, 1] \rightarrow \left[-\frac{1}{2}, \frac{1}{2}\right]$, $f(x) = \frac{x}{1+x^2}$, then find the area bounded by $y = f^{-1}(x)$, x axis and lines $x = \frac{1}{2}$, $x = -\frac{1}{2}$.

ANSWERS

- $\frac{1}{k+1} (x^4 - kx^3 + 2(k-2)x^2)$
- $\frac{x}{\sqrt{x^2 + 1}}$
- $\frac{1}{2}$
- $x^{1/n}$
- 2 sq. units
- $\frac{64}{3}$ sq. units
- $(1 - \log 2)$ sq. units

Exercises

Single Correct Answer Type

- Area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ is equal to
 - π sq. units
 - $\pi/2$ sq. units
 - $\frac{3\pi}{4}$ sq. units
 - $\frac{3\pi}{2}$ sq. units
- Let $f(x) = \min(x+1, \sqrt{1-x})$ for all $x \leq 1$. Then the area bounded by $y = f(x)$ and the x -axis is
 - $\frac{7}{3}$ sq. units
 - $\frac{1}{6}$ sq. units
 - $\frac{11}{6}$ sq. units
 - $\frac{7}{6}$ sq. units
- The area of the closed figure bounded by $x = -1$, $y = 0$, $y = x^2 + x + 1$, and the tangent to the curve $y = x^2 + x + 1$ at $A(1, 3)$ is
 - $4/3$ sq. units
 - $7/3$ sq. units
 - $7/6$ sq. units
 - None of these
- The area bounded by the curve $a^2y = x^2(x+a)$ and the x -axis is
 - $a^2/3$ sq. units
 - $a^2/4$ sq. units
 - $3a^2/4$ sq. units
 - $a^2/12$ sq. units
- The area between the curve $y = 2x^4 - x^2$, the x -axis, and the ordinates of the two minima of the curve is
 - $11/60$ sq. units
 - $7/120$ sq. units
 - $1/30$ sq. units
 - $7/90$ sq. units
- The area of the closed figure bounded by $x = -1$, $x = 2$, and $y = \begin{cases} -x^2 + 2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$ and the abscissa axis is
 - $16/3$ sq. units
 - $10/3$ sq. units
 - $13/3$ sq. units
 - $7/3$ sq. units
- The value of the parameter a such that the area bounded by $y = a^2x^2 + ax + 1$, coordinate axes, and the line $x = 1$ attains its least value is equal to
 - $-\frac{1}{4}$ sq. units
 - $-\frac{1}{2}$ sq. units
 - $-\frac{3}{4}$ sq. units
 - -1 sq. units
- The positive value of the parameter ' k ' for which the area of the figure bounded by the curve $y = \sin(kx)$, $x = \frac{2\pi}{3k}$, $x = \frac{5\pi}{3k}$ and x -axis is less than 2 can be
 - $\frac{1}{8} < k < \frac{3}{8}$
 - $0 < k < \frac{1}{8}$
 - $1 < k < 2$
 - $\frac{3}{8} < k < \frac{5}{8}$
- The area bounded by the curve $y = x(1 - \log_e x)$ and x -axis is
 - $\frac{e^2}{4}$
 - $\frac{e^2}{2}$
 - $\frac{e^2 - e}{2}$
 - $\frac{e^2 - e}{4}$
- The area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is
 - $12\sqrt{3}$ sq. units
 - $6\sqrt{3}$ sq. units
 - $8\sqrt{3}$ sq. units
 - $4\sqrt{3}$ sq. units
- Area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$ is
 - $\frac{3\pi - 8}{3}$ sq. units
 - $\frac{\pi - 8}{3}$ sq. units
 - $\frac{2\pi - 8}{3}$ sq. units
 - None of these
- If A_n is the area bounded by $y = x$ and $y = x^n$, $n \in N$, then $A_2 \cdot A_3 \dots A_n =$
 - $\frac{1}{n(n+1)}$
 - $\frac{1}{2^n n(n+1)}$
 - $\frac{1}{2^{n-1} n(n+1)}$
 - $\frac{1}{2^{n-2} n(n+1)}$
- The area of the region in 1st quadrant bounded by the y -axis, $y = \frac{x}{4}$, $y = 1 + \sqrt{x}$, and $y = \frac{2}{\sqrt{x}}$ is
 - $2/3$ sq. units
 - $8/3$ sq. units
 - $11/3$ sq. units
 - $13/6$ sq. units
- The area of the closed figure bounded by $y = \frac{x^2}{2} - 2x + 2$ and the tangents to it at $(1, 1/2)$ and $(4, 2)$ is
 - $9/8$ sq. units
 - $3/8$ sq. units
 - $3/2$ sq. units
 - $9/4$ sq. units
- The area of the region bounded by $x^2 + y^2 - 2x - 3 = 0$ and $y = |x| + 1$ is
 - $\frac{\pi}{2} - 1$ sq. units
 - 2π sq. units
 - 4π sq. units
 - $\pi/2$ sq. units
- The area enclosed by the curve $y = \sqrt{4 - x^2}$, $y \geq \sqrt{2} \sin\left(\frac{x\pi}{2\sqrt{2}}\right)$, and the x -axis is divided by the y -axis in the ratio
 - $\frac{\pi^2 - 8}{\pi^2 + 8}$
 - $\frac{\pi^2 - 4}{\pi^2 + 4}$
 - $\frac{\pi - 4}{\pi - 4}$
 - $\frac{2\pi^2}{2\pi + \pi^2 - 8}$

17. The area bounded by the curve $y^2 = 1 - x$ and the lines $y = \frac{|x|}{x}$, $x = -1$, and $x = \frac{1}{2}$ is
- (1) $\frac{3}{\sqrt{2}} - \frac{11}{6}$ sq. units (2) $3\sqrt{2} - \frac{11}{4}$ sq. units
- (3) $\frac{6}{\sqrt{2}} - \frac{11}{5}$ sq. units (4) None of these
18. The area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is
- (1) $e - 2$ sq. units (2) $3 - e$ sq. units
- (3) e sq. units (4) $e - 1$ sq. units
19. The area bounded by $y = 3 - |3 - x|$ and $y = \frac{6}{|x + 1|}$ is
- (1) $\frac{15}{2} - 6 \ln 2$ sq. units
- (2) $\frac{13}{2} - 3 \ln 2$ sq. units
- (3) $\frac{13}{2} - 6 \ln 2$ sq. units
- (4) None of these
20. The area enclosed between the curves $y = \log_e (x + e)$, $x = \log_e \left(\frac{1}{y} \right)$, and the x -axis is
- (1) 2 sq. units (2) 1 sq. units
- (3) 4 sq. units (4) None of these
21. The area enclosed between $y = \sin 2x$, $y = \sqrt{3} \sin x$ between $x = 0$ and $x = \pi$ is
- (1) $\frac{7}{2}$ sq. units (2) $\frac{7}{4} + \sqrt{3}$ sq. units
- (3) $\frac{7\sqrt{3}}{4}$ sq. units (4) $7 - \frac{\sqrt{3}}{4}$ sq. units
22. The area bounded by $y = x^2$, $y = [x + 1]$, $0 \leq x \leq 2$ and the y -axis is where $[.]$ is greatest integer function.
- (1) $\frac{1}{3}$ (2) $\frac{\sqrt{2}}{3}$
- (3) 1 (4) $\frac{7}{3}$
23. The area of the figure bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point with the ordinate $x = 3$, and the x -axis is
- (1) 7 sq. units (2) 6 sq. units
- (3) 9 sq. units (4) None of these
24. The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is
- (1) $\frac{2}{e}$ sq. units (2) $1 - \frac{2}{e}$ sq. units
- (3) $\frac{1}{e}$ sq. units (4) $1 - \frac{1}{e}$ sq. units
25. The area of the region whose boundaries are defined by the curves $y = 2 \cos x$, $y = 3 \tan x$, and the y -axis is
- (1) $1 + 3 \ln \left(\frac{2}{\sqrt{3}} \right)$ sq. units
- (2) $1 + \frac{3}{2} \ln 3 - 3 \ln 2$ sq. units
- (3) $1 + \frac{3}{2} \ln 3 - \ln 2$ sq. units
- (4) $\ln 3 - \ln 2$ sq. units
26. The area bounded by $y = \sec^{-1} x$, $y = \operatorname{cosec}^{-1} x$, and line $x - 1 = 0$ is
- (1) $\log(3 + 2\sqrt{2}) - \frac{\pi}{2}$ sq. units
- (2) $\frac{\pi}{2} - \log(3 + 2\sqrt{2})$ sq. units
- (3) $\pi - \log_e 3$ sq. units
- (4) None of these
27. The area bounded by the curve $y^2(2 - x) = x^3$ and $x = 2$ is
- (1) $\frac{\pi}{2}$ (2) π (3) 2π (4) 3π
28. The area enclosed by $y = x^2 + \cos x$ and its normal at $x = \frac{\pi}{2}$ in the first quadrant is
- (1) $\frac{\pi^5}{32} - \frac{\pi^4}{64} + \frac{\pi^3}{32} + 1$ (2) $\frac{\pi^5}{16} - \frac{\pi^4}{32} + \frac{\pi^3}{24} - 1$
- (3) $\frac{\pi^5}{32} - \frac{\pi^4}{32} + \frac{\pi^3}{16}$ (4) $\frac{\pi^5}{32} - \frac{\pi^4}{32} + \frac{\pi^3}{24} + 1$
29. Given $f(x) = \int_0^x e^t (\log_e \sec t - \sec^2 t) dt$, $g(x) = -2e^x \tan x$, then the area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = 0$ and $x = \frac{\pi}{3}$, is (in sq. units)
- (1) $\frac{1}{2} e^{\frac{\pi}{3}} \log_e 2$ (2) $e^{\frac{\pi}{3}} \log_e 2$
- (3) $\frac{1}{4} e^{\frac{\pi}{3}} \log_e 2$ (4) $e^{\frac{\pi}{3}} \log_e 3$
30. The area of the loop of the curve $ay^2 = x^2(a - x)$ is
- (1) $4a^2$ sq. units (2) $\frac{8a^2}{15}$ sq. units
- (3) $\frac{16a^2}{9}$ sq. units (4) None of these
31. The area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y| \sqrt{1 - y^2}$ is
- (1) 1 sq. units (2) $4/3$ sq. units
- (3) $2/3$ sq. units (4) 2 sq. units
32. The area bounded by the loop of the curve $4y^2 = x^2(4 - x^2)$ is
- (1) $7/3$ sq. units (2) $8/3$ sq. units
- (3) $11/3$ sq. units (4) $16/3$ sq. units

33. The area enclosed by the curves $xy^2 = a^2(a-x)$ and $(a-x)y^2 = a^2x$ is
 (1) $(\pi-2)a^2$ sq. units (2) $(4-\pi)a^2$ sq. units
 (3) $\pi a^2/3$ sq. units (4) None of these
34. The area bounded by the two branches of curve $(y-x)^2 = x^3$ and the straight line $x = 1$ is
 (1) $1/5$ sq. units (2) $3/5$ sq. units
 (3) $4/5$ sq. units (4) $8/4$ sq. units
35. The area bounded by the curves $y = \sin^{-1} |\sin x|$ and $y = (\sin^{-1} |\sin x|)^2$, where $0 \leq x \leq 2\pi$, is
 (1) $\frac{1}{3} + \frac{\pi^2}{4}$ sq. units (2) $\frac{1}{6} + \frac{\pi^3}{8}$ sq. units
 (3) 2 sq. units (4) None of these
36. Consider two curves $C_1: y^2 = 4[\sqrt{y}]x$ and $C_2: x^2 = 4[\sqrt{x}]y$, where $[\cdot]$ denotes the greatest integer function. Then the area of region enclosed by these two curves within the square formed by the lines $x=1, y=1, x=4, y=4$ is
 (1) $8/3$ sq. units (2) $10/3$ sq. units
 (3) $11/3$ sq. units (4) $11/4$ sq. units
37. The area enclosed between the curve $y^2(2a-x) = x^3$ and the line $x = 2$ above the x -axis is
 (1) πa^2 sq. units (2) $\frac{3\pi a^2}{2}$ sq. units
 (3) $2\pi a^2$ sq. units (4) $3\pi a^2$ sq. units
38. The area of the region of the plane bounded by $\max(|x|, |y|) \leq 1$ and $xy \leq \frac{1}{2}$ is
 (1) $1/2 + \ln 2$ sq. units (2) $3 + \ln 2$ sq. units
 (3) $31/4$ sq. units (4) $1 + 2 \ln 2$ sq. units
39. The area of the region containing the points (x, y) satisfying $4 \leq x^2 + y^2 \leq 2(|x| + |y|)$ is
 (1) 8 sq. units (2) 2 sq. units
 (3) 4π sq. units (4) 2π sq. units
40. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, the x -axis, and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2}\beta$. Then $f'\left(\frac{\pi}{2}\right)$ is
 (1) $\left(\frac{\pi}{2} - \sqrt{2} - 1\right)$ (2) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$
 (3) $-\frac{\pi}{2}$ (4) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$
- (2) The range of $A(k)$ is $\left[\frac{20\sqrt{5}}{3}, \infty\right)$
 (3) If function $k \rightarrow A(k)$ is defined for $k \in [-2, \infty)$, then $A(k)$ is many-one function
 (4) The value of k for which area is minimum is 1
2. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4, y = 4$ and the coordinate axes. If S_1, S_2, S_3 are the areas of these parts numbered from top to bottom, respectively, then
 (1) $S_1:S_2 \equiv 1:1$ (2) $S_2:S_3 \equiv 1:2$
 (3) $S_1:S_3 \equiv 1:1$ (4) $S_1:(S_1 + S_2) = 1:2$
3. Which of the following have the same bounded area
 (1) $f(x) = \sin x, g(x) = \sin^2 x$, where $0 \leq x \leq 10\pi$
 (2) $f(x) = \sin x, g(x) = |\sin x|$, where $0 \leq x \leq 20\pi$
 (3) $f(x) = |\sin x|, g(x) = \sin^3 x$, where $0 \leq x \leq 10\pi$
 (4) $f(x) = \sin x, g(x) = \sin^4 x$, where $0 \leq x \leq 10\pi$
4. If the curve $y = ax^{1/2} + bx$ passes through the point $(1, 2)$ and lies above the x -axis for $0 \leq x \leq 9$ and the area enclosed by the curve, the x -axis, and the line $x = 4$ is 8 sq. units, then
 (1) $a = 1$ (2) $b = 1$ (3) $a = 3$ (4) $b = -1$
5. The area enclosed by the curves $x = a \sin^3 t$ and $y = a \cos^3 t$ is equal to
 (1) $12a^2 \int_0^{\pi/2} \cos^4 t \sin^2 t dt$
 (2) $12a \int_0^{\pi/2} \cos^2 t \sin^4 t dt$
 (3) $2 \int_{-a}^a (a^{2/3} - x^{2/3})^{3/2} dx$
 (4) $4 \int_0^a (a^{2/3} - x^{2/3})^{3/2} dx$
6. If A_i is the area bounded by $|x - a_i| + |y| = b_i, i \in N$, where $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}, a_1 = 0, b_1 = 32$, then
 (1) $A_3 = 128$
 (2) $A_3 = 256$
 (3) $\lim_{n \rightarrow \infty} \sum_{i=1}^n A_i = \frac{8}{3}(32)^2$
 (4) $\lim_{n \rightarrow \infty} \sum_{i=1}^n A_i = \frac{4}{3}(16)^2$
7. Area of the region bounded by the curve $y = e^{x^2}$ and the line $y = e$ is
 (1) $2 \int_1^e \sqrt{\log_e y} dy$ (2) $2e - \int_{-1}^1 e^{x^2} dx$
 (3) $\int_{-1}^1 (e - e^{x^2}) dx$ (4) $2 \int_0^1 \sqrt{x} e^x dx$

Multiple Correct Answers Type

1. Let $A(k)$ be the area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$.
 (1) The range of $A(k)$ is $\left[\frac{10\sqrt{5}}{3}, \infty\right)$

8. Consider curves $y = \frac{1}{x^2}$; $y = \frac{1}{4(x-1)}$. Let α be the value of a ($a > 2$) for which area bounded by curves between $x = 2$ and $x = a$ is $1/a$ is $e^2 + 1$ and β be the of $b \in (1, 2)$, for which the area bounded by curves between $x = b$ and $x = 2$ is $1 - \frac{1}{b}$, then

- (1) $\alpha = e^2 + 1$ (2) $\alpha = e^2 - 2$
 (3) $\beta = 1 + e^{-1}$ (4) $\beta = 1 + e^{-2}$

9. Consider curves $S_1: \sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$, $S_2: x^2 + y^2 = a^2$ and $S_3: |x| + |y| = a$. If α is area bounded by S_1 and S_2 , β is area bounded by S_1 and S_3 and γ is the area bounded by S_2 and S_3 , then

(1) $\alpha = a^2 \left(\pi - \frac{2}{3} \right)$

(2) $\beta = \frac{4a^2}{3}$

(3) $\gamma = 2a^2 (\pi - 1)$

- (4) the ratio in which S_3 divides area between S_1 and S_2 is $4 : 3(\pi - 2)$

10. Let $A(k)$ be the area bounded by the curves $y = x^2 + 2x - 3$ and $y = kx + 1$. Then

- (1) the value of k for which $A(k)$ is least is 2
 (2) the value of k for which $A(k)$ is least is $3/2$
 (3) least value of $A(k)$ is $32/3$
 (4) least value of $A(k)$ is $64/3$

Linked Comprehension Type

For Problems 1 and 2

Let A_r be the area of the region bounded between the curves $y^2 = (e^{-kr})x$ (where $k > 0$, $r \in \mathbb{N}$) and the line $y = mx$ (where $m \neq 0$), k and m are some constants.

1. $A_1, A_2, A_3, \dots$ are in G.P. with common ratio
 (1) e^{-k} (2) e^{-2k}
 (3) e^{-4k} (4) None of these

2. $\lim_{n \rightarrow \infty} \sum_{i=1}^n A_i = \frac{1}{48(e^{2k} - 1)}$, then the value of m is
 (1) 3 (2) 1 (3) 2 (4) 4

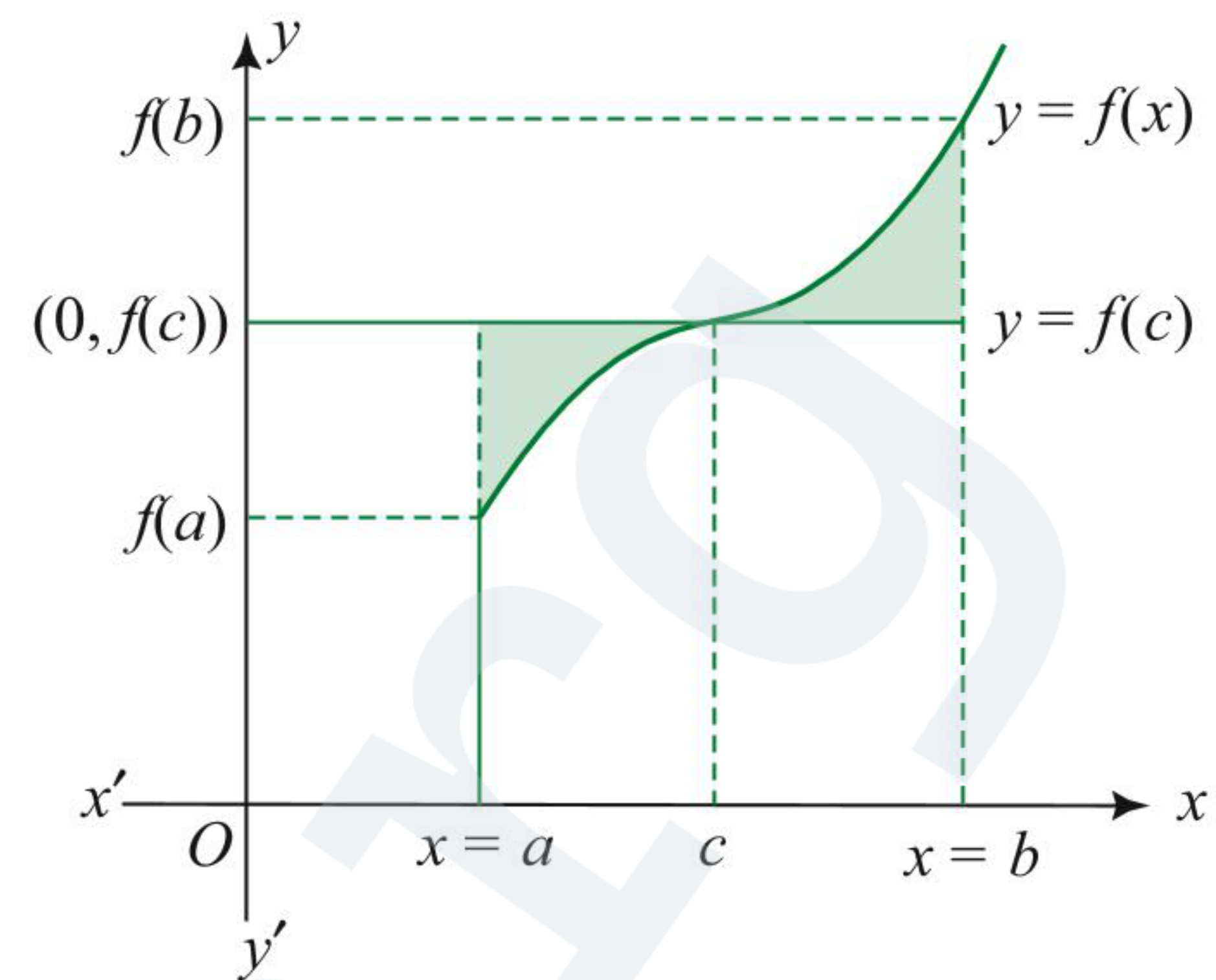
For Problems 3–5

If $y = f(x)$ is a monotonic function in (a, b) , then the area bounded by the ordinates at

$x = a$, $x = b$, $y = f(x)$ and $y = f(c)$ (where $c \in (a, b)$) is minimum when $c = \frac{a+b}{2}$.

Proof: $A = \int_a^c (f(c) - f(x)) dx + \int_c^b (f(x) - f(c)) dx$
 $= f(c)(c-a) - \int_a^c f(x) dx + \int_c^b f(x) dx - f(c)(b-c)$

$$\Rightarrow A = [2c - (a+b)]f(c) + \int_c^b f(x) dx - \int_a^c f(x) dx$$



Differentiating w.r.t. c , we get

$$\frac{dA}{dc} = [2c - (a+b)]f'(c) + 2f(c) + 0 - f(c) - (f(c) - 0)$$

For maxima and minima, $\frac{dA}{dc} = 0$

$$\Rightarrow f'(c) [2c - (a+b)] = 0 \text{ (as } f'(c) \neq 0)$$

Hence, $c = \frac{a+b}{2}$

Also for $c < \frac{a+b}{2}$, $\frac{dA}{dc} < 0$ and for $c > \frac{a+b}{2}$, $\frac{dA}{dc} > 0$

Hence, A is minimum when $c = \frac{a+b}{2}$.

3. If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight

lines $x = 0$, $x = 2$, and the x -axis is minimum, then the value of a is

- (1) $1/3$ (2) 2 (3) 1 (4) $2/3$

4. The value of the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$, and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, which is the least, is

- (1) 2 (2) 0 (3) -1 (4) 1

5. If the area enclosed by $f(x) = \sin x + \cos x$, $y = a$ between two consecutive points of extremum is minimum, then the value of a is

- (1) 0 (2) -1 (3) 1 (4) 2

For Problems 6–8

Consider the areas $S_0, S_1, S_2 \dots$ bounded by the x -axis and half-waves of the curve $y = e^{-x} \sin x$, where $x \geq 0$.

6. The value of S_0 is

- (1) $\frac{1}{2}(1 + e^\pi)$ sq. units (2) $\frac{1}{2}(1 + e^{-\pi})$ sq. units

- (3) $\frac{1}{2}(1 - e^{-\pi})$ sq. units (4) $\frac{1}{2}(e^\pi - 1)$ sq. units

7. The sequence $S_0, S_1, S_2, \dots$, forms a G.P. with common ratio

(1) $\frac{e^\pi}{2}$ (2) $e^{-\pi}$ (3) e^π (4) $\frac{e^{-\pi}}{2}$

8. $\sum_{n=0}^{\infty} S_n$ is equal to

(1) $\frac{1+e^\pi}{1-e^{-\pi}}$ (2) $\frac{\frac{1}{2}(1+e^\pi)}{1-e^{-\pi}}$
 (3) $\frac{1}{2(1-e^{-\pi})}$ (4) None of these

For Problems 9–11

Two curves $C_1 \equiv [f(y)]^{2/3} + [f(x)]^{1/3} = 0$ and $C_2 \equiv [f(y)]^{2/3} + [f(x)]^{2/3} = 12$, satisfying the relation $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$.

9. The area bounded by C_1 and C_2 is
 (1) $2\pi - \sqrt{3}$ sq. units (2) $2\pi + \sqrt{3}$ sq. units
 (3) $\pi + \sqrt{6}$ sq. units (4) $2\sqrt{3} - \pi$ sq. units
10. The area bounded by the curve C_2 and $|x| + |y| = \sqrt{12}$ is
 (1) $12\pi - 24$ sq. units (2) $6 - \sqrt{12}$ sq. units
 (3) $2\sqrt{12} - 6$ sq. units (4) None of these
11. The area bounded by C_1 and $x + y + 2 = 0$ is
 (1) $5/2$ sq. units (2) $7/2$ sq. units
 (3) $9/2$ sq. units (4) None of these

For Problems 12–13

Consider the two curves $C_1: y = 1 + \cos x$ and $C_2: y = 1 + \cos(x - \alpha)$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$, where $x \in [0, \pi]$. Also the area of the figure bounded by the curves C_1, C_2 , and $x = 0$ is same as that of the figure bounded by $C_2, y = 1$, and $x = \pi$.

12. The value of α is
 (1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{8}$
13. For the values of α , the area bounded by $C_1, C_2, x = 0$, and $x = \pi$ is
 (1) 1 sq. units (2) 2 sq. units
 (3) $2 + \sqrt{3}$ sq. units (4) None of these

For Problems 14–16

Consider the function defined implicitly by the equation $y^2 - 2ye^{\sin^{-1} x} + x^2 - 1 + [x] + e^{2\sin^{-1} x} = 0$ (where $[x]$ denotes the greatest integer function).

14. The area of the region bounded by the curve and the line $x = -1$ is
 (1) $\pi + 1$ sq. units (2) $\pi - 1$ sq. units
 (3) $\frac{\pi}{2} + 1$ sq. units (4) $\frac{\pi}{2} - 1$ sq. units
15. Line $x = 0$ divides the region mentioned above in two parts. The ratio of area of left-hand side of line to that of right-hand side of line is
 (1) $1 + \pi : \pi$ (2) $2 - \pi : \pi$
 (3) $1:1$ (4) $\pi + 2 : \pi$

16. The area of the region of curve and line $x = 0$ and $x = \frac{1}{2}$ is

(1) $\frac{\sqrt{3}}{4} + \frac{\pi}{6}$ sq. units (2) $\frac{\sqrt{3}}{2} + \frac{\pi}{6}$ sq. units
 (3) $\frac{\sqrt{3}}{4} - \frac{\pi}{6}$ sq. units (4) $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ sq. units

For Problems 17–19

Computing area with parametrically represented boundaries:

If the boundary of a figure is represented by parametric equation, i.e., $x = x(t), y = y(t)$, then the area of the figure is evaluated by one of the three formulas:

$$S = - \int_{\alpha}^{\beta} y(t) x'(t) dt,$$

$$S = \int_{\alpha}^{\beta} x(t) y'(t) dt,$$

$$S = \frac{1}{2} \int_{\alpha}^{\beta} (xy' - yx') dt,$$

where α and β are the values of the parameter t corresponding respectively to the beginning and the end of the traversal of the curve corresponding to increasing t .

17. The area of the region bounded by an arc of the cycloid $x = a(t - \sin t), y = a(1 - \cos t)$ and the x -axis is
 (1) $6\pi a^2$ sq. units (2) $3\pi a^2$ sq. units
 (3) $4\pi a^2$ sq. units (4) None of these
18. The area of the loop described as $x = \frac{t}{3}(6-t), y = \frac{t^2}{8}(6-t)$ is
 (1) $\frac{27}{5}$ sq. units (2) $\frac{24}{5}$ sq. units
 (3) $\frac{27}{6}$ sq. units (4) $\frac{21}{5}$ sq. units
19. If the curve given by parametric equation $x = t - t^3, y = 1 - t^4$ forms a loop for all values of $t \in [-1, 1]$, then the area of the loop is
 (1) $\frac{1}{7}$ sq. units (2) $\frac{3}{5}$ sq. units
 (3) $\frac{16}{35}$ sq. units (4) $\frac{8}{35}$ sq. units

For Problems 20 and 21

Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \leq 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}, \text{ then}$$

20. If $f(x)$ is continuous for all real x then the value of $a^2 + b^2$ is
 (1) 3 (2) 4 (3) 5 (4) 11

21. The area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$ is
- (1) sq. units

(2) $\frac{257}{192}$ sq. units

(3) $\frac{257}{96}$ sq. units

(4) $\frac{289}{192}$ sq. units

Matrix Match Type

1. Match the following lists:

List I	List II
a. The area bounded by the curve $y = x x $, x -axis and the ordinates $x = 1, x = -1$	p. $10/3$ sq. units
b. The area of the region lying between the lines $x - y + 2 = 0, x = 0$, and the curve $x = \sqrt{y}$	q. $64/3$ sq. units
c. The area enclosed between the curves $y^2 = x$ and $y = x $	r. $2/3$ sq. units
d. The area bounded by parabola $y^2 = x$, straight line $y = 4$, and the y -axis	s. $1/6$ sq. units

2. Match the following lists:

List I	List II
a. Area enclosed by $y = [x]$ and $y = \{x\}$, where $[.]$ and $\{.\}$ represent greatest integer and fractional part functions, respectively	p. $32/5$ sq. units
b. The area bounded by the curves $y^2 = x^3$ and $ y = 2x$.	q. 1 sq. units
c. The smaller area included between the curves $\sqrt{x} + \sqrt{ y } = 1$ and $ x + y = 1$.	r. 4 sq. units
d. Area bounded by the curves $y = \left[\frac{x^2}{64} + 2\right]$ (where $[.]$ denotes the greatest integer function), $y = x - 1$ and $x = 0$ above the x -axis.	s. $2/3$ sq. units

3. Match the following lists:

List I: [.] represents greatest integer	List II
a. Area enclosed by $[x]^2 = [y]^2$ for $1 \leq x \leq 4$	p. 8 sq. units
b. Area enclosed by $[x] + [y] = 2$	q. 6 sq. units
c. Area enclosed by $[x] [y] = 2$	r. 4 sq. units
d. Area enclosed by $\frac{[x]}{[y]} = 2, -5 \leq x \leq 5$	s. 12 sq. units

4. Match the following lists:

List I	List II
a. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$. Then the value of $[a^2]$ is, where $[.]$ represents greatest integer part function	p. 1
b. The area bounded by the curves $f(x) = \begin{cases} x^{\frac{1}{\log_e x}}, & x \neq 1 \\ e, & x = 1 \end{cases}$ and $y = x - e $ is A then the value of $[A]$ is, where $[.]$ represents greatest integer part function	q. 6
c. Area bounded by the curves $y = e^x, y = \log_e x$ and the lines $x = 0, y = 0, y = 1$ is A , then the value of $[A]$ is, where $[.]$ represents greatest integer part function	r. 8
d. The area of the region whose boundaries are defined by the curves $y = 2 \cos x, y = 3 \tan x$ and the y -axis, is $1 + \frac{3}{2} \log_e 3 - \log_e k$, then the value of k is	s. 7

Codes:

- a

b

c

d
- (1)

(2)

(3)

(4)
- q

p

s

p
- s

p

s

p
- r

q

p

r
- r

r

r

s

Numerical Value Type

1.

The area enclosed by the curve $C: y = x\sqrt{9 - x^2}$ ($x \geq 0$) and the x -axis is ____.
2.

Let S be the area bounded by the curve $y = \sin x$ ($0 \leq x \leq \pi$) and the x -axis and T be the area bounded by the curves $y = \sin x \left(0 \leq x \leq \frac{\pi}{2}\right), y = a \cos x \left(0 \leq x \leq \frac{\pi}{2}\right)$, and the x -axis (where $a \in R^+$). The value of $(3a)$ such that $S: T = 1 : \frac{1}{3}$ is ____.
3.

Let C be a curve passing through $M(2, 2)$ such that the slope of the tangent at any point to the curve is reciprocal of the ordinate of the point. If the area bounded by curve C and line $x = 2$ is A , then the value of $\frac{3A}{2}$ is ____.
4.

The area enclosed by $f(x) = 12 + ax + -x^2$ coordinates axes and the ordinates at $x = 3$ ($f(3) > 0$) is 45 sq. units. If m and n are the x -axis intercepts of the graph of $y = f(x)$, then the value of $(m + n + a)$ is ____.
5.

If the area bounded by the curve $f(x) = x^{1/3}(x - 1)$ and the x -axis is A , then the value of $28A$ is ____.
6.

If the area bounded by the curve $y = x^2 + 1$ and the tangents to it drawn from the origin is A , then the value of $3A$ is ____.

7. If the area enclosed by the curve $y = \sqrt{x}$ and $x = -\sqrt{y}$, the circle $x^2 + y^2 = 2$ above the x -axis is A , then the value of $\frac{16}{\pi}A$ is ____.
8. The value of a ($a > 0$) for which the area bounded by the curves $y = \frac{x}{6} + \frac{1}{x^2}$, $y = 0$, $x = a$, and $x = 2a$ has the least value is ____.
9. Area bounded by the relation $[2x] + [y] = 5$, $x, y > 0$ is ____ (where $[\cdot]$ represents greatest integer function).
10. The area bounded by the curves $y = x(x-3)^2$ and $y = x$ is ____ (in sq. units).
11. If the area of the region $\{(x, y): 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\}$ is A , then the value of $3A - 17$ is ____.
12. If S is the sum of possible values of c for which the area of the figure bounded by the curves $y = \sin 2x$, the straight lines $x = \pi/6$, $x = c$, and the abscissa axis is equal to $1/2$, then the value of π/S is ____.
13. If A is the area bounded by the curves $y = \sqrt{1-x^2}$ and $y = x^3 - x$, then the value of π/A is ____.
14. Consider two curves $C_1: y = \frac{1}{x}$ and $C_2: y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1 , C_2 , and the line $x = 1$ and D_2 denotes the region surrounded by C_1 , C_2 , and the line $x = a$. If $D_1 = D_2$, then the sum of logarithm of possible values of a is ____.
15. If ' a ' ($a > 0$) is the value of parameter for each of which the area of the figure bounded by the straight line $y = \frac{a^2 - ax}{1 + a^4}$ and the parabola $y = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$ is the greatest, then the value of a^4 is ____.
16. If S is the sum of cubes of possible value of c for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, then straight lines $x = 1$ and $x = c$ and the abscissa axis is equal to $16/3$, then the value of $[S]$, where $[\cdot]$ denotes the greatest integer function, is ____.

Archives

JEE ADVANCED

Single Correct Answer Type

1. Let the straight line $x = b$ divide the area enclosed by $y = (1-x)^2$, $y = 0$, and $x = 0$ into two parts R_1 ($0 \leq x \leq b$) and R_2 ($b \leq x \leq 1$) such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(1) $3/4$ (2) $1/2$ (3) $1/3$ (4) $1/4$

(IIT-JEE 2011)

2. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $[0, \pi/2]$ is

(1) $4(\sqrt{2} - 1)$ (2) $2\sqrt{2}(\sqrt{2} - 1)$
(3) $2(\sqrt{2} + 1)$ (4) $2\sqrt{2}(\sqrt{2} + 1)$

(JEE Advanced 2013)

3. Area of the region

$\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq x+9 \leq 15\}$ is equal to

(1) $\frac{1}{6}$ (2) $\frac{4}{3}$ (3) $\frac{3}{2}$ (4) $\frac{5}{3}$

(JEE Advanced 2016)

4. The area of the region $\{(x, y) : xy \leq 8, 1 \leq y \leq x^2\}$ is

(1) $16 \log_e 2 - 6$ (2) $8 \log_e 2 - \frac{7}{2}$
(3) $16 \log_e 2 - \frac{14}{3}$ (4) $8 \log_e 2 - \frac{14}{3}$

(JEE Advanced 2019)

5. Let the functions $f: R \rightarrow R$ and $g: R \rightarrow R$ be defined by

$$f(x) = e^{x-1} - e^{-|x-1|} \text{ and } g(x) = \frac{1}{2}(e^{x-1} + e^{1-x}).$$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

(1) $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$

(2) $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$

(3) $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

(4) $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$ (JEE Advanced 2020)

Multiple Correct Answers Type

1. The area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(1) $e - 1$ (2) $\int_1^e \ln(e+1-y) dy$

(3) $e - \int_0^1 e^x dx$ (4) $\int_1^e \ln y dy$

(IIT-JEE 2009)

2. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\}$ into two equal parts, then

- (1) $\frac{1}{2} < \alpha < 1$ (2) $\alpha^4 + 4\alpha^2 - 1 = 0$
 (3) $0 < \alpha \leq \frac{1}{2}$ (4) $2\alpha^4 - 4\alpha^2 + 1 = 0$

(JEE Advanced 2017)

3. Let $f: [0, \infty) \rightarrow R$ be a continuous function such that

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t) dt \text{ for all } x \in [0, \infty). \text{ Then,}$$

which of the following statement(s) is (are) TRUE?

- (1) The curve $y = f(x)$ passes through the point (1, 2)
 (2) The curve $y = f(x)$ passes through the point (2, -1)
 (3) The area of the region

$$\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1-x^2}\} \text{ is } \frac{\pi-2}{4}$$

- (4) The area of the region

$$\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1-x^2}\} \text{ is } \frac{\pi-1}{4}$$

(JEE Advanced 2018)

Matrix Match Type

1. Match the statements given in List I with the values given in List II.

List I	List II
a. In a triangle ΔXYZ , let a , b and c be the lengths of the sides opposite to the angles X , Y and Z , respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	p. 1
b. In a triangle ΔXYZ , let a , b and c be the lengths of the sides opposite to the angles X , Y and Z , respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	q. 2

c. In R^2 , let $\sqrt{3}\hat{i} + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X , Y and Z with respect of the origin O , respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $|\beta|$ is (are)

r. 3

d. Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)

s. 5

t. 6

(JEE Advanced 2015)

Numerical Value Type

1. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is _____.

(JEE Advanced 2014)

2. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighboring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ ($n > 1$). If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR , then the value of n is _____.

(JEE Advanced 2018)

Answers Key**EXERCISES****Single Correct Answer Type**

1. (1) 2. (4) 3. (3) 4. (4) 5. (2)
 6. (1) 7. (3) 8. (3) 9. (1) 10. (1)
 11. (1) 12. (4) 13. (3) 14. (1) 15. (1)
 16. (4) 17. (1) 18. (2) 19. (3) 20. (1)
 21. (1) 22. (2) 23. (3) 24. (1) 25. (2)
 26. (1) 27. (4) 28. (4) 29. (2) 30. (2)
 31. (4) 32. (4) 33. (1) 34. (3) 35. (4)
 36. (3) 37. (2) 38. (2) 39. (1) 40. (3)

Multiple Correct Answers Type

1. (2), (3) 2. (1), (3), (4)
 3. (1), (3), (4) 4. (3), (4)
 5. (1), (3), (4) 6. (1), (3)
 7. (1), (2), (3), (4) 8. (1), (4)
 9. (1), (2), (4) 10. (1), (3)

Linked Comprehension Type

1. (2) 2. (3) 3. (4) 4. (3) 5. (1)
 6. (1) 7. (3) 8. (2) 9. (2) 10. (1)
 11. (3) 12. (3) 13. (2) 14. (1) 15. (4)

16. (1)17. (2)18. (1)19. (3)20. (3)
21. (2)

Matrix Match Type

1. $a \rightarrow r$; $b \rightarrow p$; $c \rightarrow s$; $d \rightarrow q$
2. $a \rightarrow q$; $b \rightarrow p$; $c \rightarrow s$; $d \rightarrow r$
3. $a \rightarrow q$; $b \rightarrow s$; $c \rightarrow p$; $d \rightarrow p$
4. (3)

Numerical Value Type

1. (9)2. (4)3. (8)4. (8)5. (9)
6. (2)7. (8)8. (1)9. (3)10. (8)
11. (6)12. (6)13. (2)14. (1)15. (3)
16. (2)

ARCHIVES

JEE ADVANCED

Single Correct Answer Type

1. (2)2. (2)3. (3)4. (3)5. (1)

Multiple Correct Answers Type

1. (2), (3), (4)2. (1), (4)
3. (2), (3)

Matrix Match Type

1. $d \rightarrow s, t$

Numerical Value Type

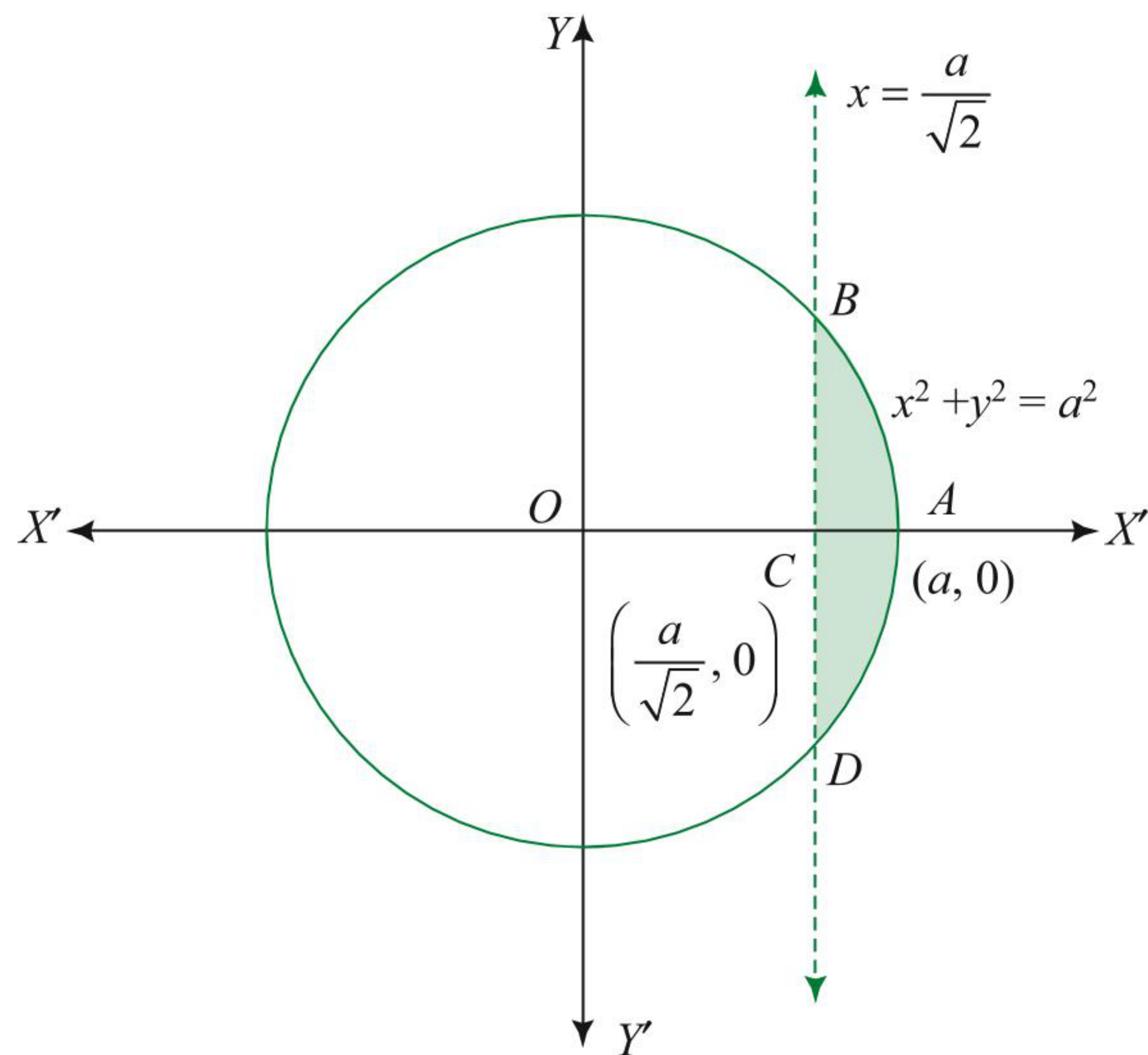
1. (6)2. (4)

Chapter 9

Concept Application Exercises

Exercise 9.1

1. The area of the smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is the area $ABCD$.



Solving $x^2 + y^2 = a^2$ and $x = \frac{a}{\sqrt{2}}$ for their points of intersection, we get

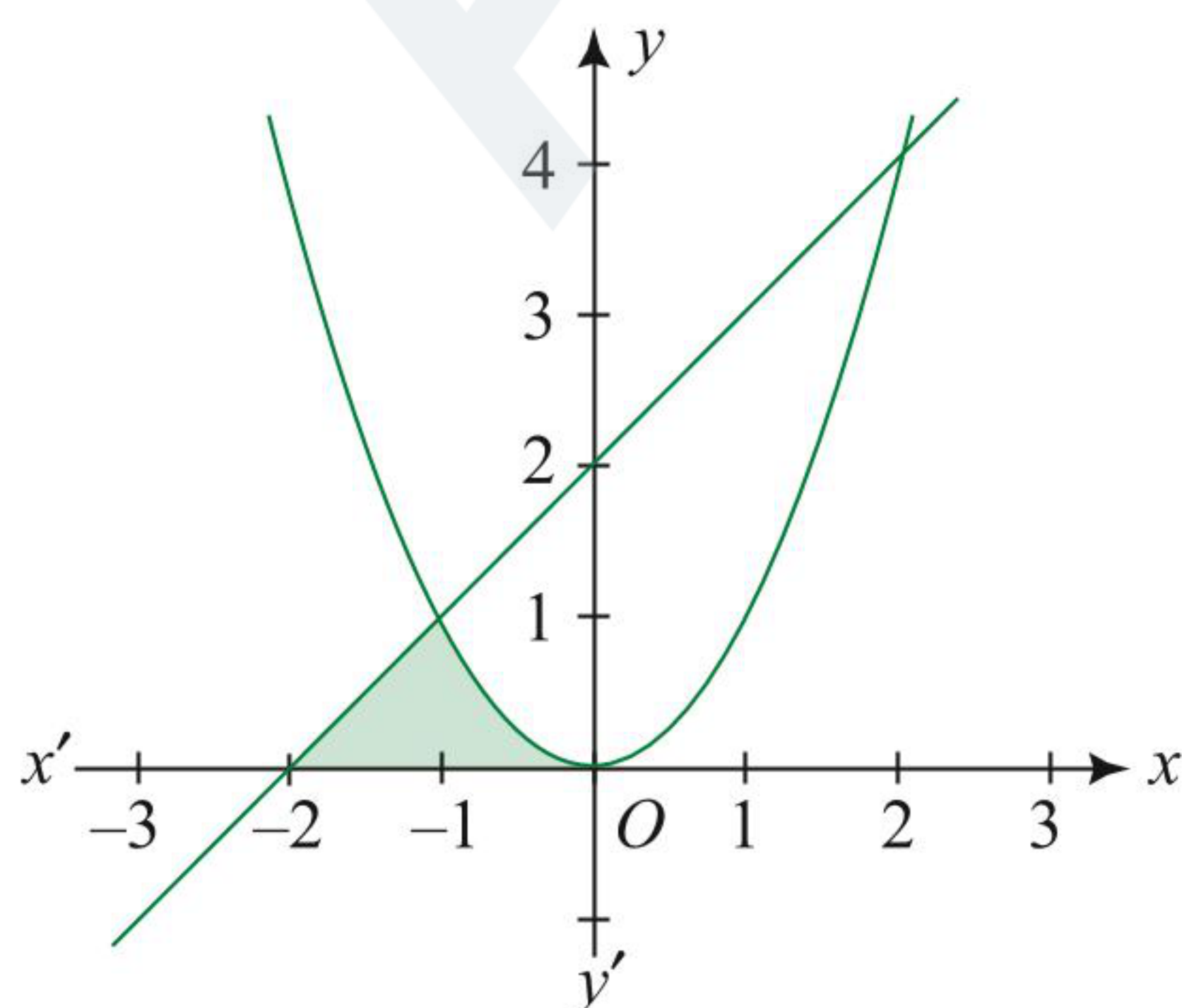
$$\frac{a^2}{2} + y^2 = a^2$$

or $y^2 = \frac{a^2}{2}$ or $y = \pm \frac{a}{\sqrt{2}}$

$\therefore$ Area $ABCD = 2 \times \text{Area } ABC$

$$\begin{aligned} &= 2 \int_{\frac{a}{\sqrt{2}}}^a \sqrt{a^2 - x^2} dx = 2 \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_{\frac{a}{\sqrt{2}}}^a \\ &= 2 \left[\frac{a^2}{2} \left(\frac{\pi}{2} \right) - \frac{a}{2\sqrt{2}} \sqrt{a^2 - \frac{a^2}{2}} - \frac{a^2}{2} \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) \right] \\ &= 2 \left[\frac{a^2\pi}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8} \right] \\ &= \frac{a^2}{2} \left(\frac{\pi}{2} - 1 \right) \text{ sq. units.} \end{aligned}$$

2.



$$\begin{aligned} \text{Required area} &= \int_{-2}^{-1} (x+2) dx + \int_{-1}^0 x^2 dx \\ &= \left[\frac{x^2}{2} + 2x \right]_{-2}^{-1} + \left[\frac{x^3}{3} \right]_{-1}^0 \\ &= \left(\frac{1}{2} - 2 \right) - (2 - 4) + \left(0 + \frac{1}{3} \right) \\ &= \frac{5}{6} \text{ sq. units.} \end{aligned}$$

3. The given curve is

$$y = \begin{cases} \sqrt{4-x^2}, & 0 \leq x < 1 \\ \sqrt{3x}, & 1 \leq x \leq 3 \end{cases}$$

Obviously, the curve is the arc of the circle $x^2 + y^2 = 4$ (1)

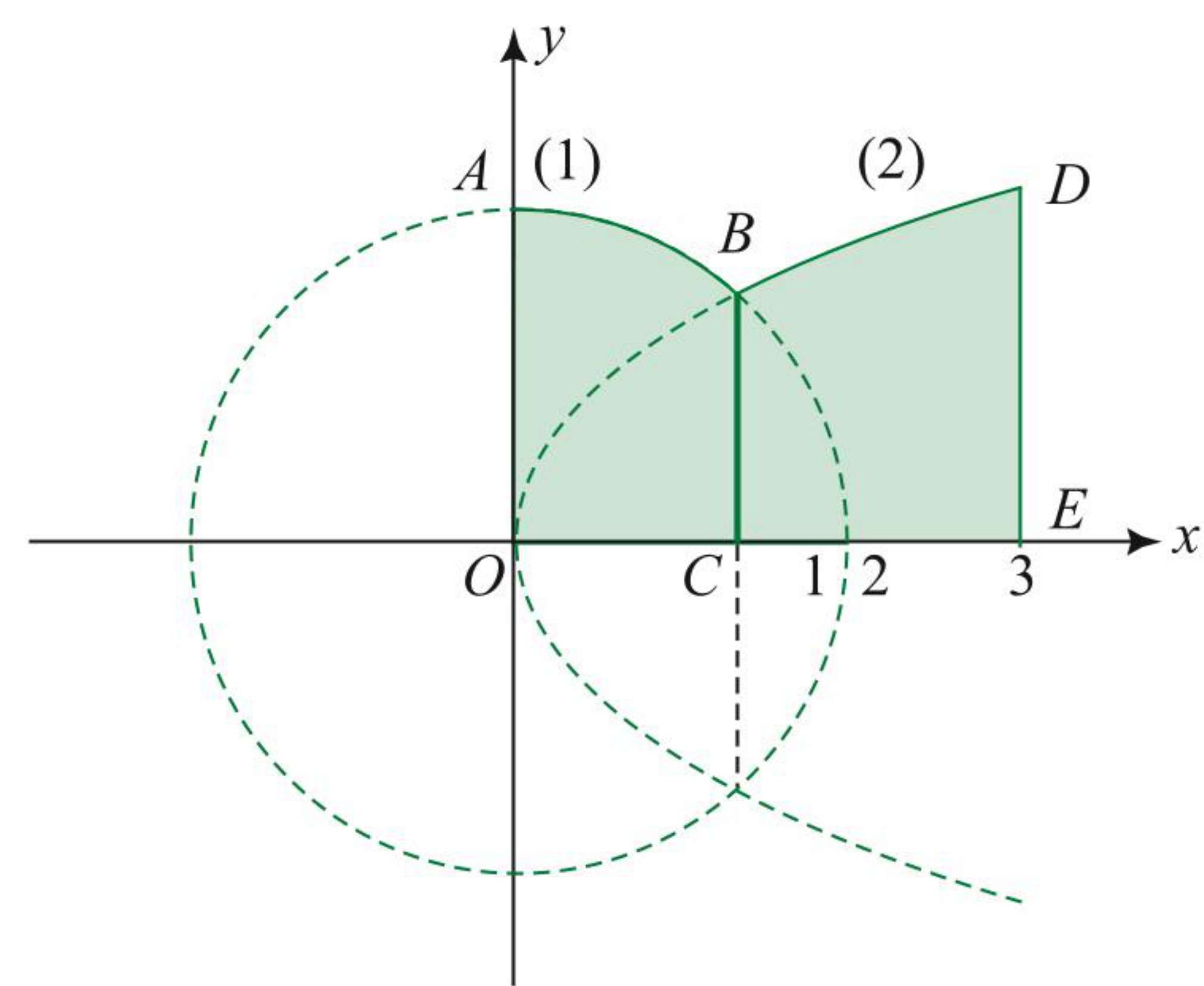
between $0 \leq x < 1$ and the arc of parabola $y^2 = 3x$ (2)

between $1 \leq x \leq 3$

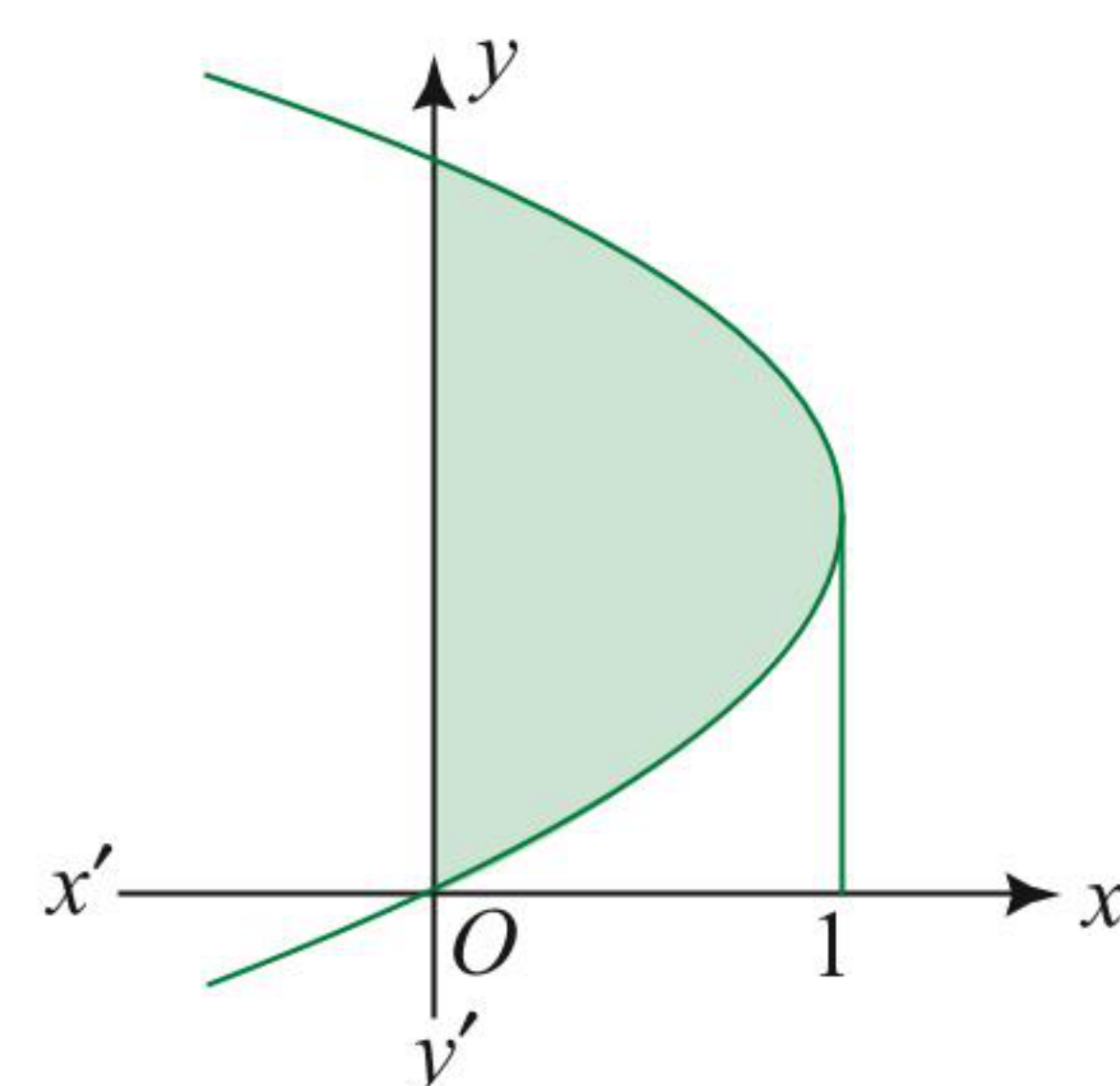
Required area = Shaded area

$$= \text{Area } OABCO + \text{Area } CBDEC$$

$$\begin{aligned} &= \left| \int_0^1 \sqrt{4-x^2} dx \right| + \left| \int_1^3 \sqrt{3x} dx \right| \\ &= \left| \left[\frac{1}{2} x \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \left(\frac{x}{2} \right) \right]_0^1 + \left[\sqrt{3} \left[\frac{2}{3} x^{3/2} \right]_1^3 \right] \right| \\ &= \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) + \frac{2}{3} (9 - \sqrt{3}) \\ &= \frac{1}{6} (2\pi - \sqrt{3} + 36) \text{ sq. units.} \end{aligned}$$



4.



$$A = \int_0^2 x dy = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{y^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

5. Given $y = 1 + \frac{8}{x^2}$.

Here y is always positive; hence, curve is lying above the x -axis.

$$\begin{aligned}\therefore \text{Req. area} &= \int_2^4 y \, dx = \int_2^4 \left(1 + \frac{8}{x^2}\right) dx \\ &= \left[x - \frac{8}{x}\right]_2^4 = 4.\end{aligned}$$

If $x = a$ bisects the area, then we have

$$\int_2^a \left(1 + \frac{8}{x^2}\right) dx = \left[x - \frac{8}{x}\right]_2^a = \left[a - \frac{8}{a} - 2 + 4\right] = \frac{4}{2}.$$

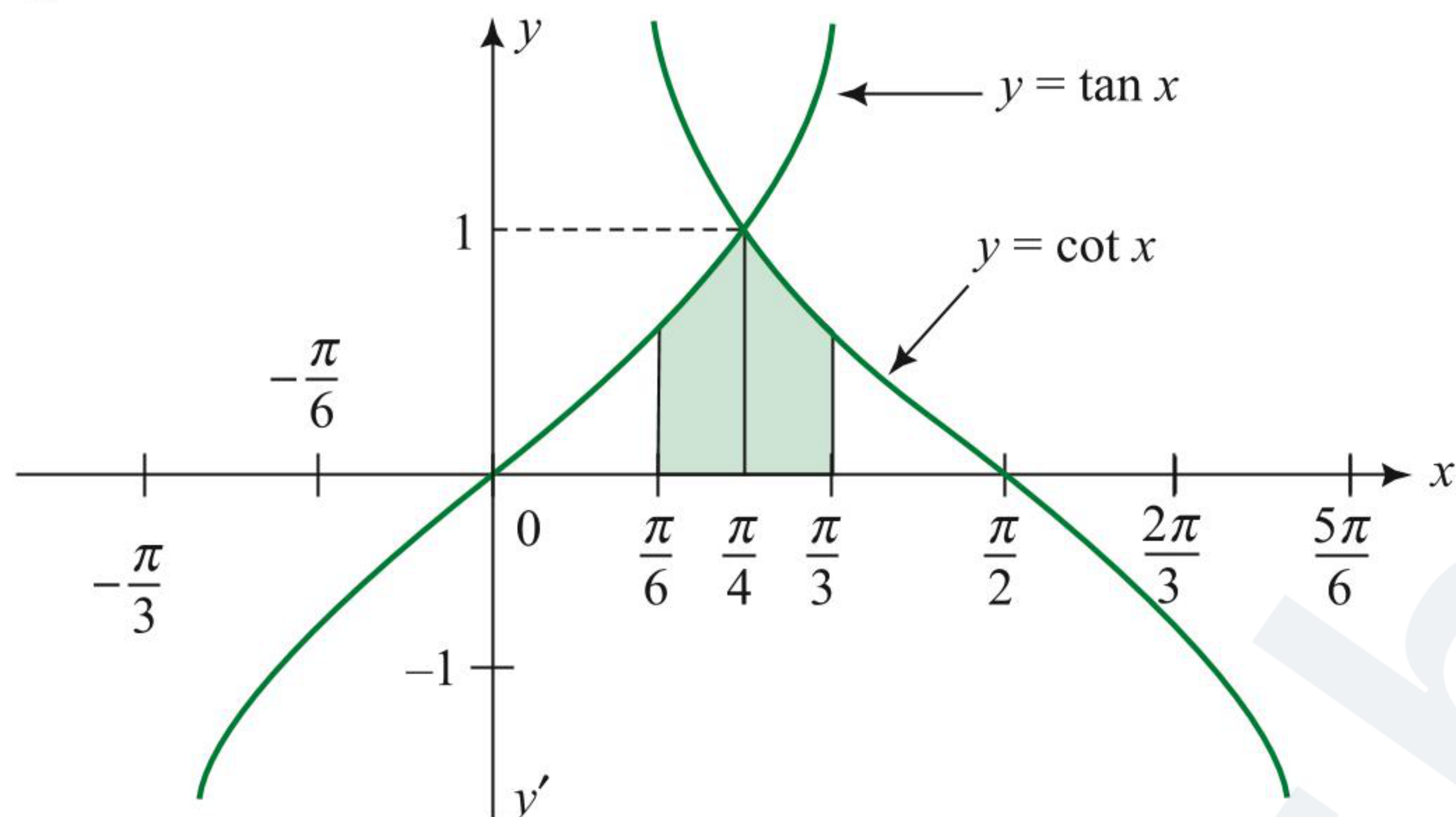
or $a - \frac{8}{a} = 0$

or $a^2 = 8$

or $a = \pm 2\sqrt{2}$

Since $a > 2$, $a = 2\sqrt{2}$.

6.



The two curves are

$$y = \tan x, \text{ where } -\pi/3 \leq x \leq \pi/3 \quad (1)$$

$$y = \cot x, \text{ where } \pi/6 \leq x \leq 3\pi/2 \quad (2)$$

At the point of intersection of the two curves

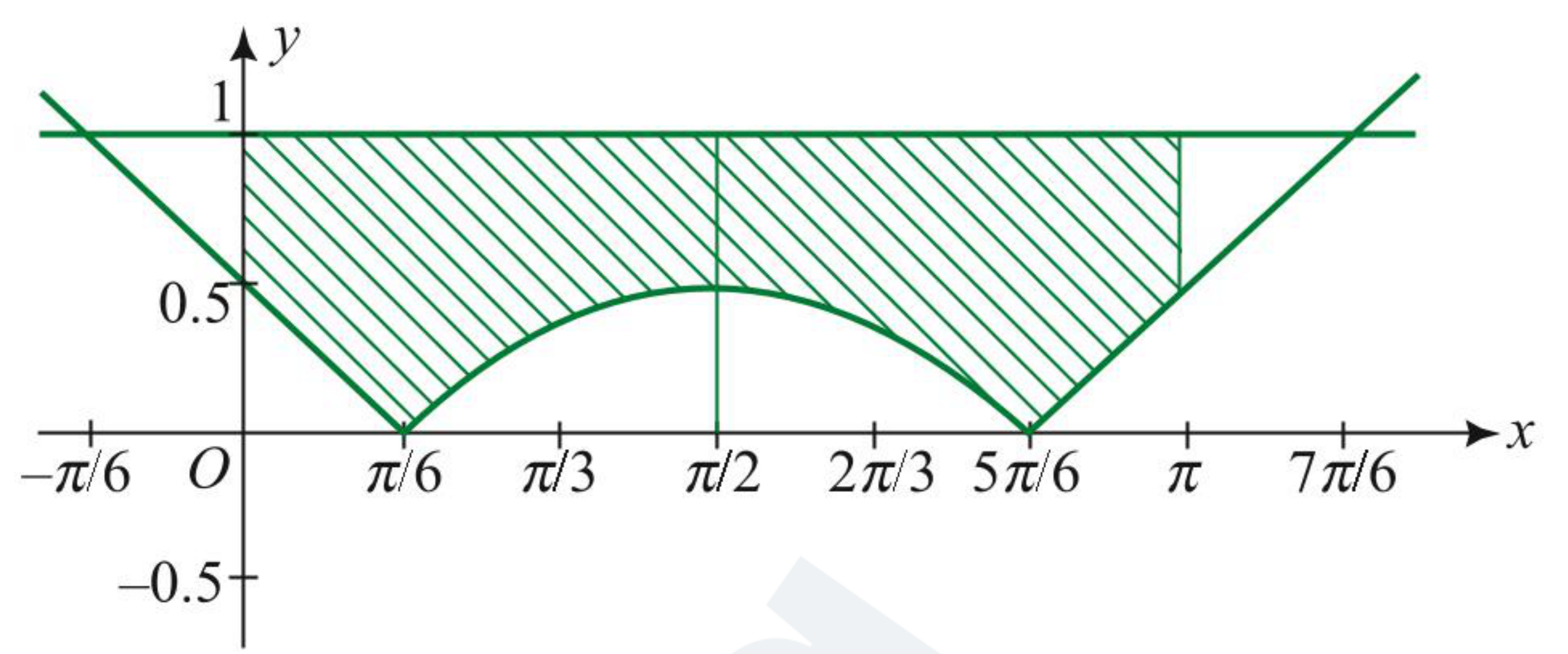
$$\tan x = \cot x \text{ or } \tan^2 x = 1 \text{ or } \tan x = \pm 1, x = \pm \pi/4$$

Thus, the curves intersect at $x = \pi/4$

The required area is the shaded area. Therefore,

$$\begin{aligned}A &= \int_{\pi/6}^{\pi/4} \tan x \, dx + \int_{\pi/4}^{\pi/3} \cot x \, dx \\ &= [\log \sec x]_{\pi/6}^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/3} \\ &= \left(\log \sqrt{2} - \log \frac{2}{\sqrt{3}}\right) + \left(\log \frac{\sqrt{3}}{2} - \log \frac{1}{\sqrt{2}}\right) \\ &= \log \sqrt{2} + \log \frac{\sqrt{3}}{2} + \log \frac{\sqrt{3}}{2} + \log \sqrt{2} \\ &= 2 \left(\log \sqrt{2} \frac{\sqrt{3}}{2}\right) \\ &= 2 \log \sqrt{\frac{3}{2}} = \log 3/2 \text{ sq. units.}\end{aligned}$$

$$7. y = \left| \sin x - \frac{1}{2} \right| = \begin{cases} \frac{1}{2} - \sin x, & 0 \leq x \leq \frac{\pi}{6} \\ \sin x - \frac{1}{2}, & \frac{\pi}{6} < x < \frac{5\pi}{6} \\ \frac{1}{2} - \sin x, & \frac{5\pi}{6} \leq x \leq \pi \end{cases}$$



$\therefore$ Required area is

$$\begin{aligned}A &= \pi - 2 \left(\int_0^{\pi/6} \left(\frac{1}{2} - \sin x\right) dx + \int_{\pi/6}^{\pi/2} \left(\sin x - \frac{1}{2}\right) dx \right) \\ &= \pi - 2 \left(\left(\frac{1}{2}x + \cos x\right)_0^{\pi/6} + 2 \left(-\cos x - \frac{1}{2}x\right)_{\pi/6}^{\pi/2} \right) \\ &= \frac{\pi}{2} + 2\end{aligned}$$

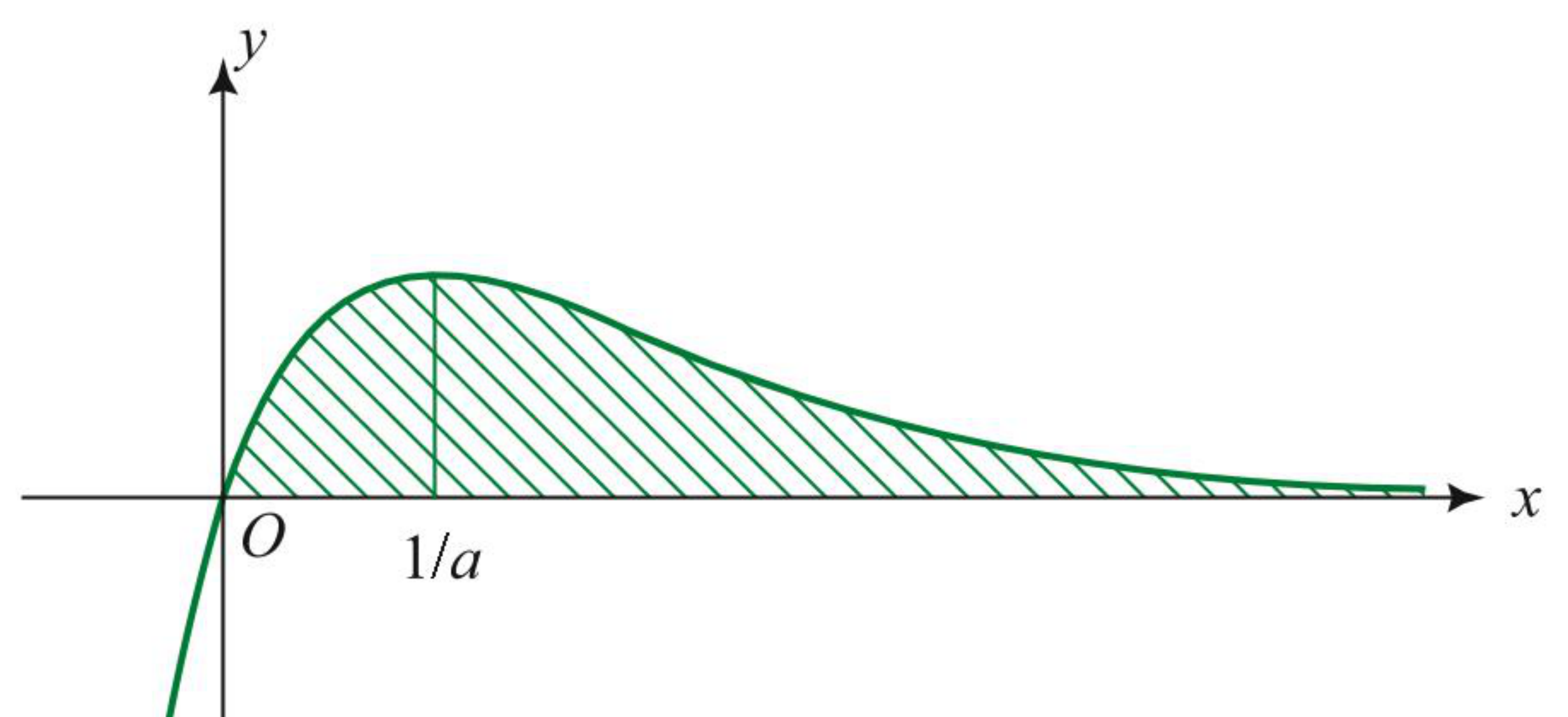
8. $y = xe^{-ax}$
 $\therefore y' = e^{-ax} - xae^{-ax}$
 $= e^{-ax}(1 - ax)$

Let $y' = 0$

$\therefore x = \frac{1}{a}$, which is point of maxima.

$$\lim_{x \rightarrow \infty} x \cdot e^{-ax} = \lim_{x \rightarrow \infty} \frac{x}{e^{ax}} = \lim_{x \rightarrow \infty} \frac{1}{ae^{ax}} = 0$$

$$\lim_{x \rightarrow -\infty} x \cdot e^{-ax} = -\infty$$



$$\begin{aligned}\text{Area} &= \int_0^{\infty} xe^{-ax} dx = \left[\frac{x \cdot e^{-ax}}{-a} \right]_0^{\infty} + \frac{1}{a} \int_0^{\infty} e^{-ax} dx \\ &= (0) - \left[\frac{1}{a^2} e^{-ax} \right]_0^{\infty} \\ &= -\frac{1}{a^2} [0 - 1] = \frac{1}{a^2}\end{aligned}$$

Given that $\frac{1}{a^2} = \frac{1}{9}$

$\Rightarrow a = 3$

9. We have curve $xy^2 = 4(2 - x)$

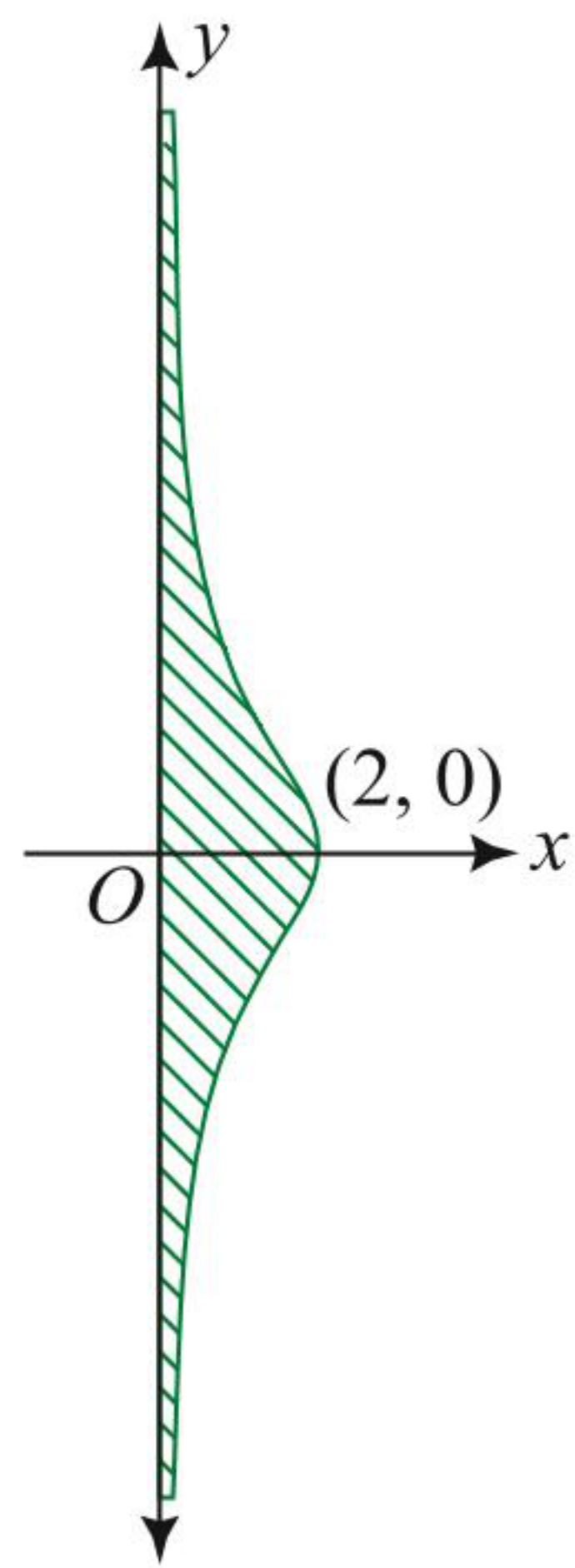
$$\therefore x = \frac{8}{y^2 + 4}$$

The given curve is symmetrical about x -axis and meets it at $(2, 0)$.

Also, when $y \rightarrow \pm \infty$, $x \rightarrow 0$.

So, y -axis is asymptote to the curve.

The graph of the function is as shown in the following figure.



The line $x = 0$, i.e., y -axis is asymptote.

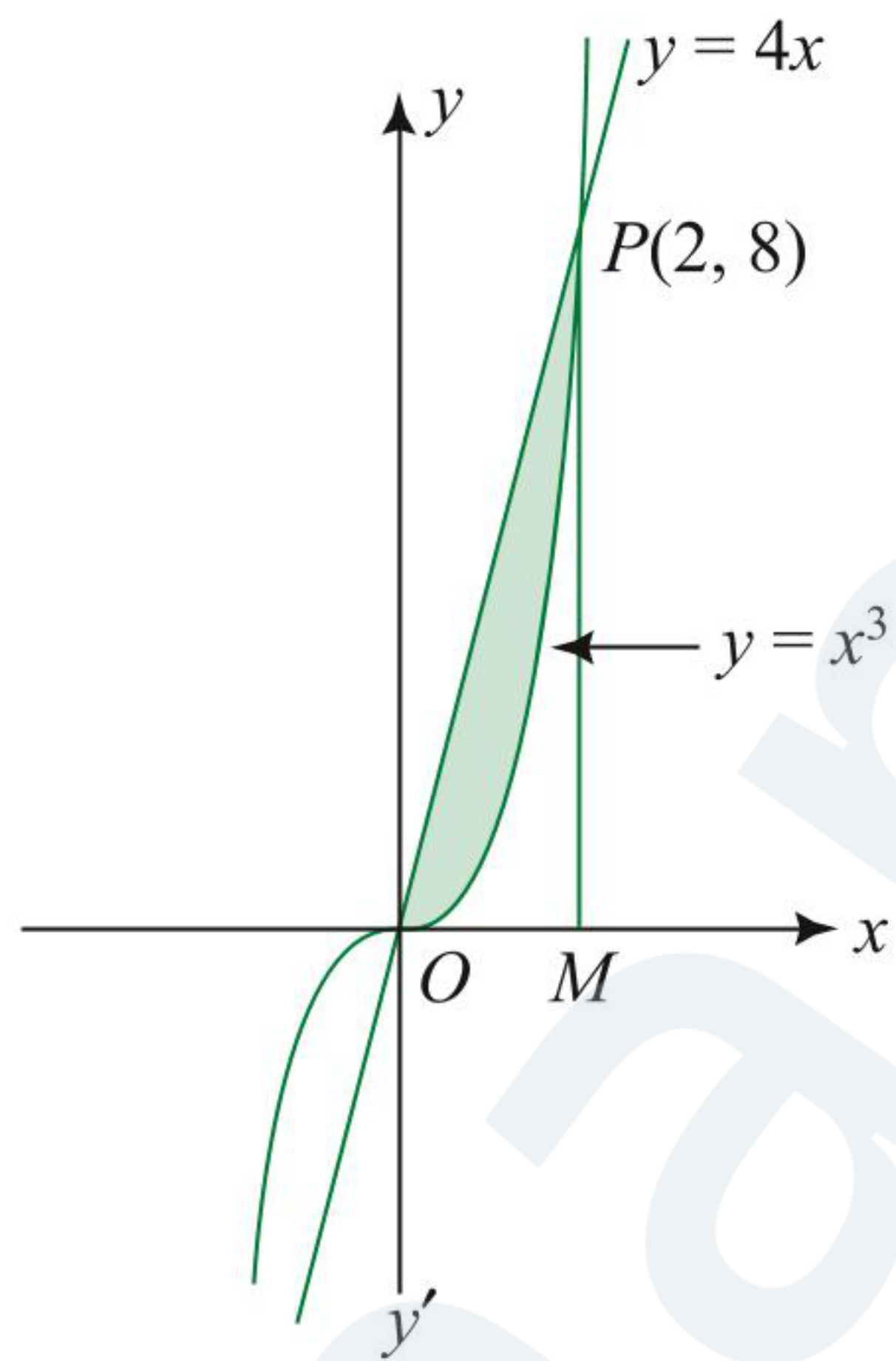
$$\begin{aligned}\text{Area of the shaded region} &= 2 \int_0^\infty \frac{8}{y^2 + 4} dx \\ &= 16 \left[\frac{1}{2} \tan^{-1} \frac{y}{2} \right]_0^\infty = 8 \times \frac{\pi}{2} \\ &= 4\pi \text{ sq. units}\end{aligned}$$

Exercise 9.2

1. The line $y = 4x$ meets $y = x^3$ at $4x = x^3$.

$$\therefore x = 0, 2, -2 \Rightarrow y = 0, 8, -8$$

$$\Rightarrow A = \int_0^2 (4x - x^3) dx = \left(2x^2 - \frac{x^4}{4} \right)_0^2 = 4 \text{ sq. units}$$



2. Given curves $x^2 = 4y$ and $x = 4y - 2$ intersect, when

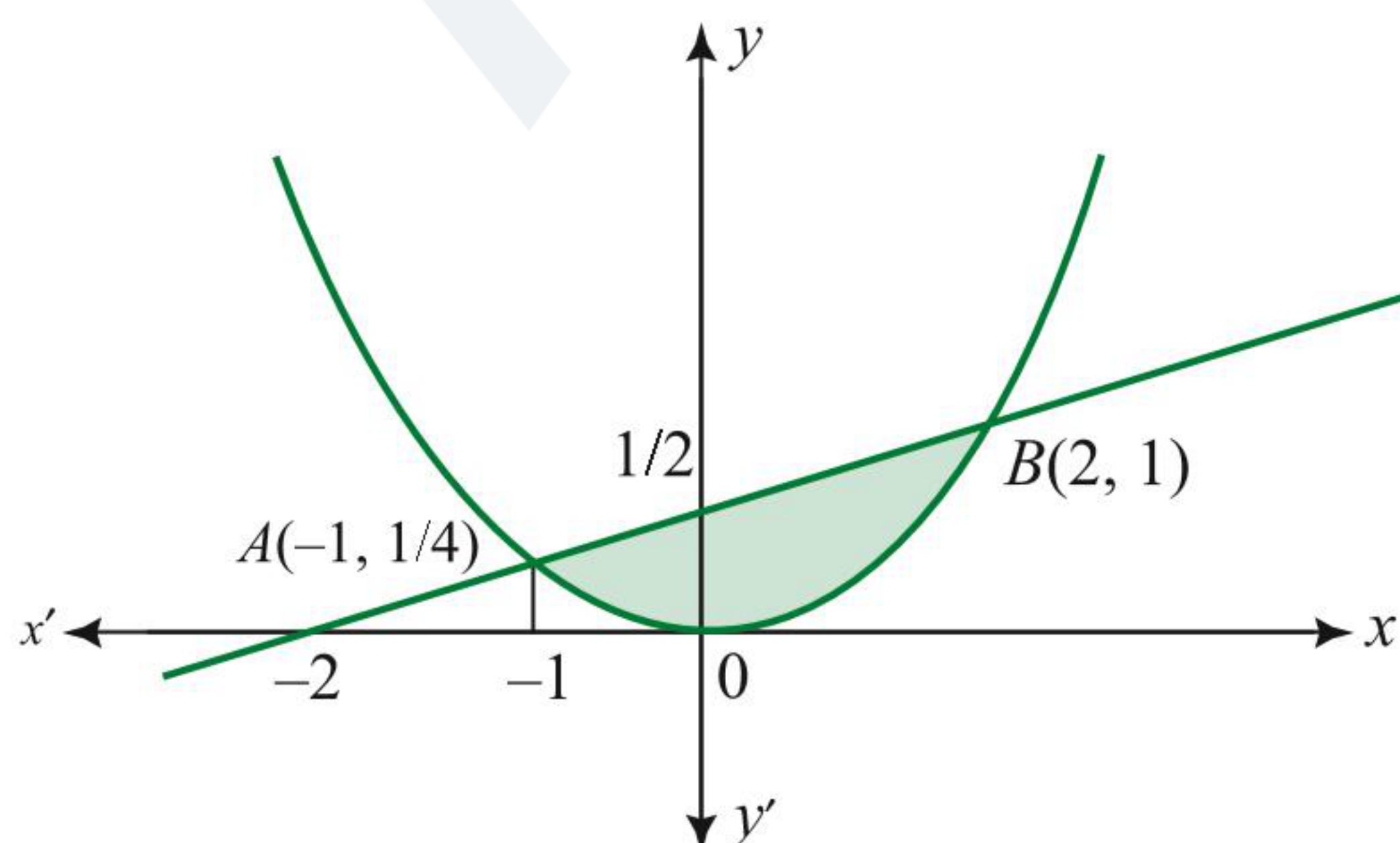
$$x^2 = x + 2$$

$$\text{or } x^2 - x - 2 = 0$$

$$\text{or } x = 2, -1$$

$$\Rightarrow y = 1, 1/4$$

Hence, points of intersection are $A(-1, 1/4)$, $B(2, 1)$



Required area = Shaded region in the figure

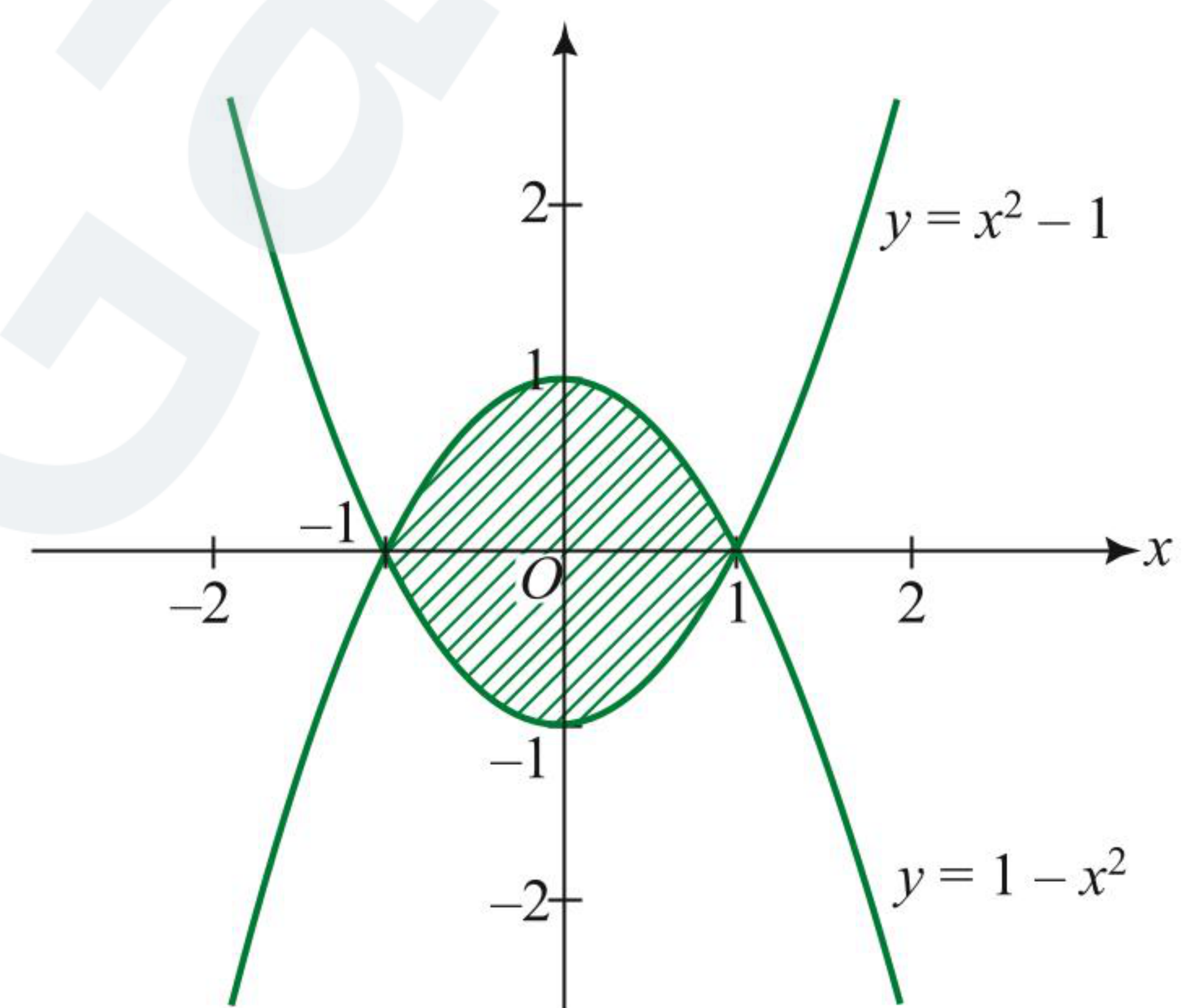
$$\begin{aligned}&= \int_{-1}^2 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx \\ &= \frac{1}{4} \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 \\ &= \frac{1}{4} \left[\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right] \\ &= \frac{1}{4} \left[\frac{10}{3} - \left(-\frac{7}{6} \right) \right] = \frac{1}{4} \left[\frac{27}{6} \right] = 9/8 \text{ sq. units.}\end{aligned}$$

3. $x^4 + 1 = 2x^2 + y^2$

$$\therefore x^4 - 2x^2 + 1 = y^2$$

$$\therefore (x^2 - 1)^2 = y^2$$

$$\therefore y = \pm(x^2 - 1), \text{ which are two parabolas.}$$



$$\begin{aligned}\text{Required area } A &= 4 \int_0^1 (1 - x^2) dx \\ &= 4 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3} \text{ sq. units}\end{aligned}$$

4. Both the given curves are parabola.

$$y = 4x - x^2 \quad (1)$$

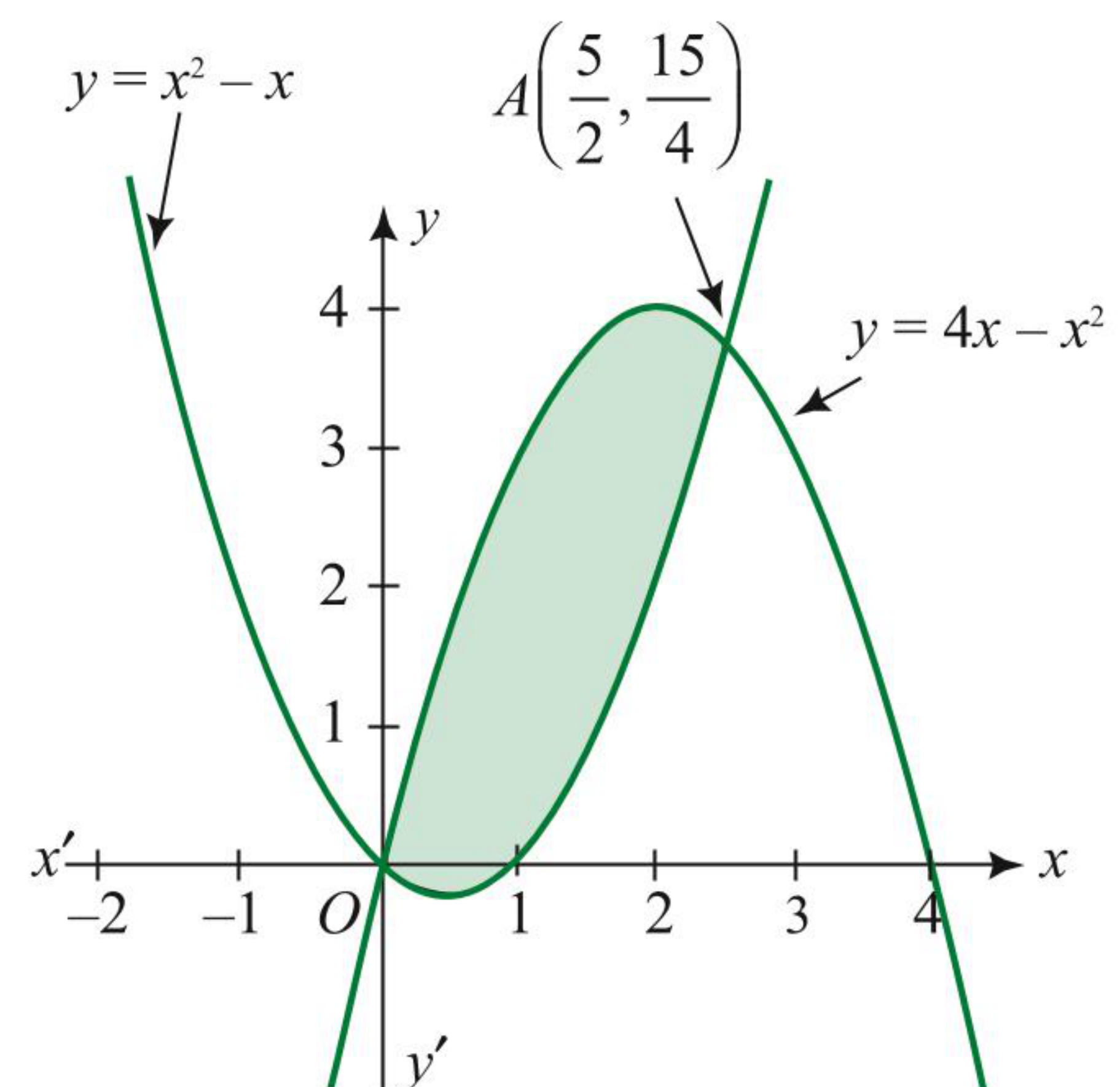
$$\text{and } y = x^2 - x \quad (2)$$

Solving (1) and (2), we get

$$4x - x^2 = x^2 - x$$

$$\text{or } x = 0, x = \frac{5}{2}.$$

Thus, two curves intersect at $O(0, 0)$ and $A\left(\frac{5}{2}, \frac{15}{4}\right)$



Here the area below x -axis,

$$\begin{aligned} A_1 &= \int_0^1 (x - x^2) dx \\ &= \left(\frac{x^2}{2} - \frac{x^3}{3} \right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units} \end{aligned}$$

Area above x -axis,

$$\begin{aligned} A_2 &= \int_0^{5/2} (4x - x^2) dx - \int_1^{5/2} (x^2 - x) dx \\ &= \left(2x^2 - \frac{x^3}{3} \right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2} \right)_1^{5/2} \\ &= \left(\frac{25}{2} - \frac{125}{24} \right) - \left[\left(\frac{125}{24} - \frac{25}{8} \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right] \\ &= \frac{25}{2} - \frac{125}{12} + \frac{25}{8} - \frac{1}{6} \\ &= \frac{300 - 250 + 75 - 4}{24} = \frac{121}{24} \text{ sq. units.} \end{aligned}$$

Thus, the ratio of area above x -axis to area below x -axis is

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4.$$

5. The given curves are

$$x^2 + y^2 = 16 \quad \dots(i)$$

$$y^2 = 6x \quad \dots(ii)$$

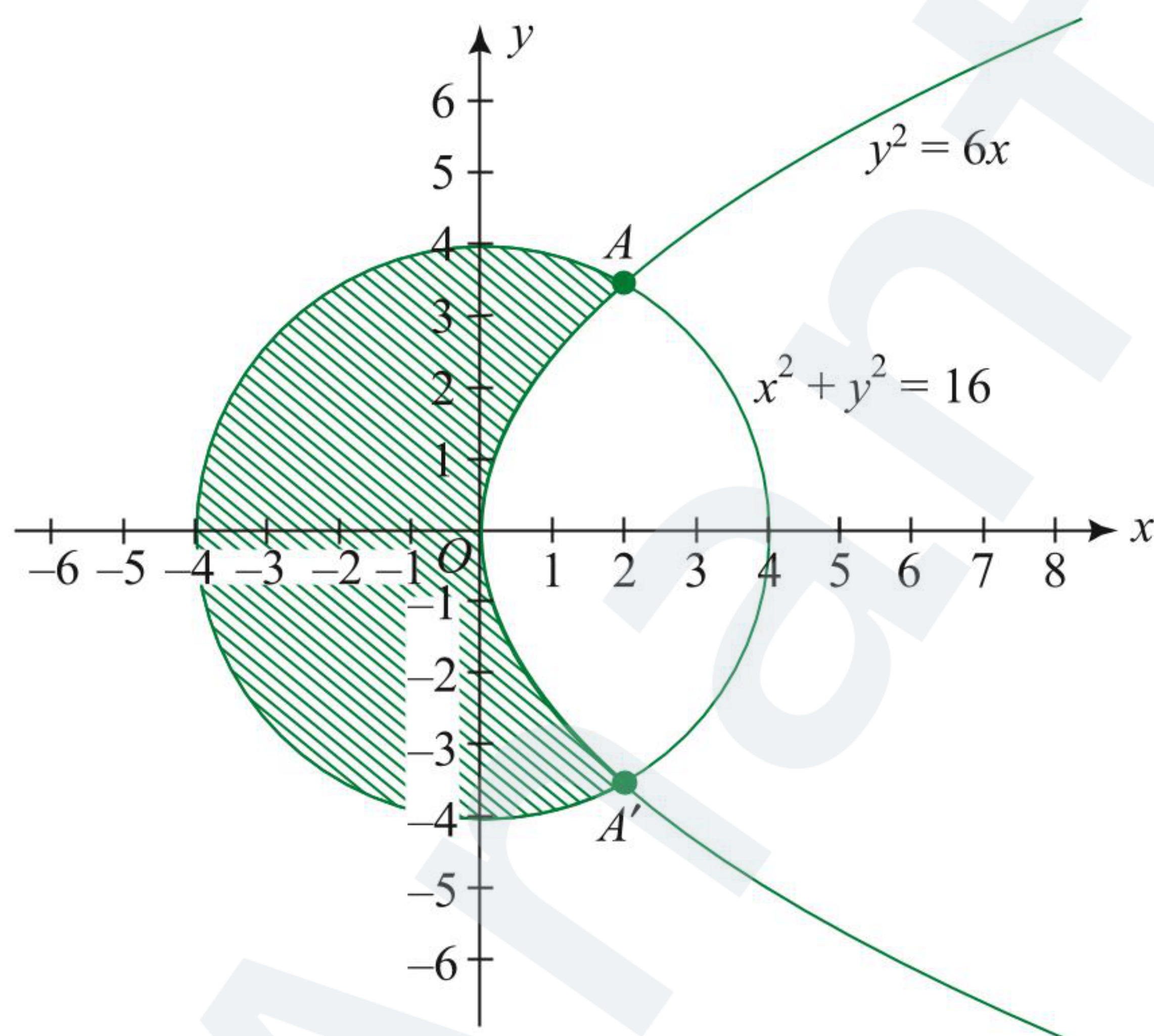
Solving $x^2 + 6x = 16$

$$\text{or } (x - 2)(x + 8) = 0$$

$$\therefore x = 2 \text{ (as } x = -8 \text{ is not possible)}$$

Thus, the points of intersection are $A(2, 2\sqrt{3})$ and $A'(2, -2\sqrt{3})$.

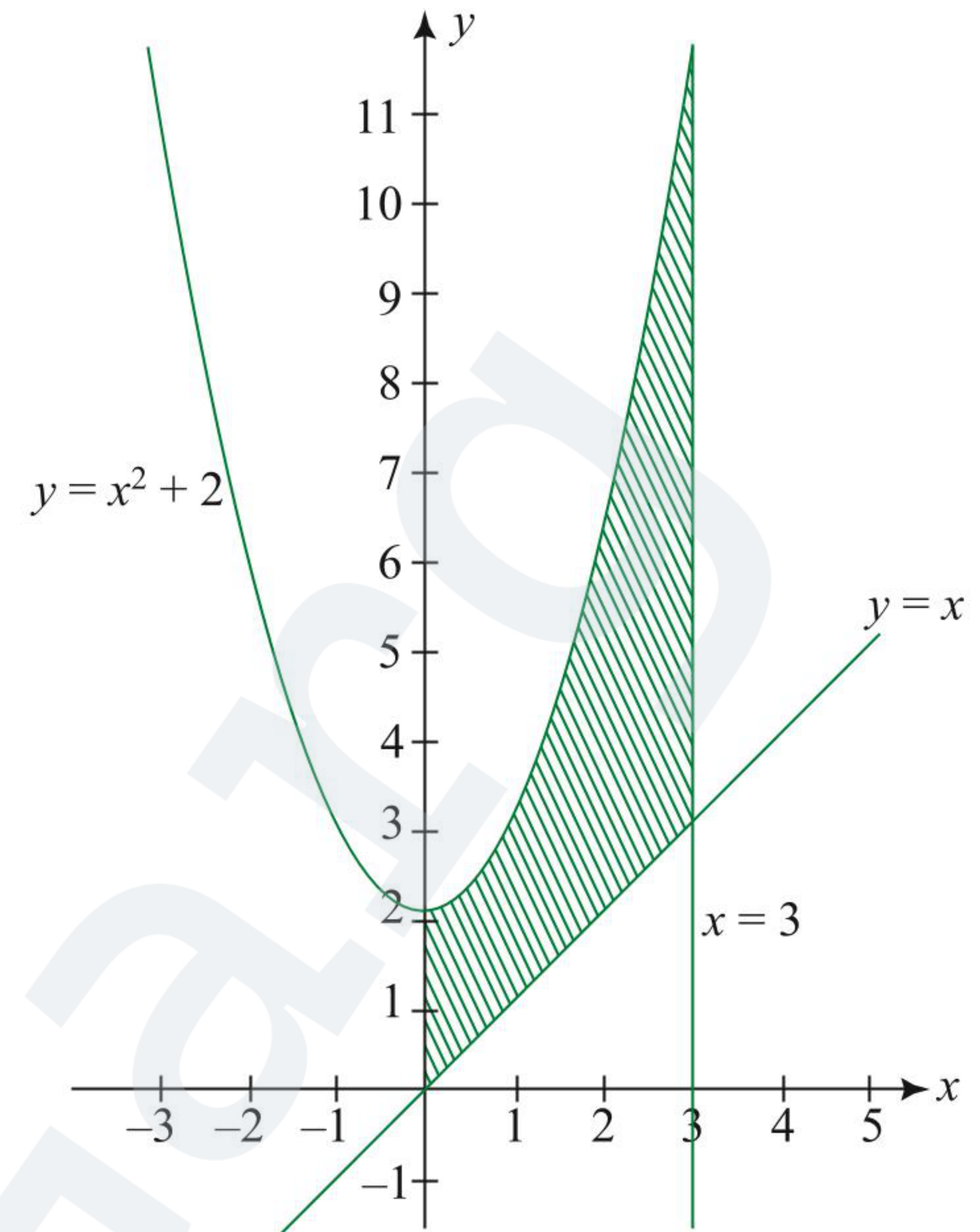
The graphs of the curves is as shown in the following figure.



From the figure, required area is

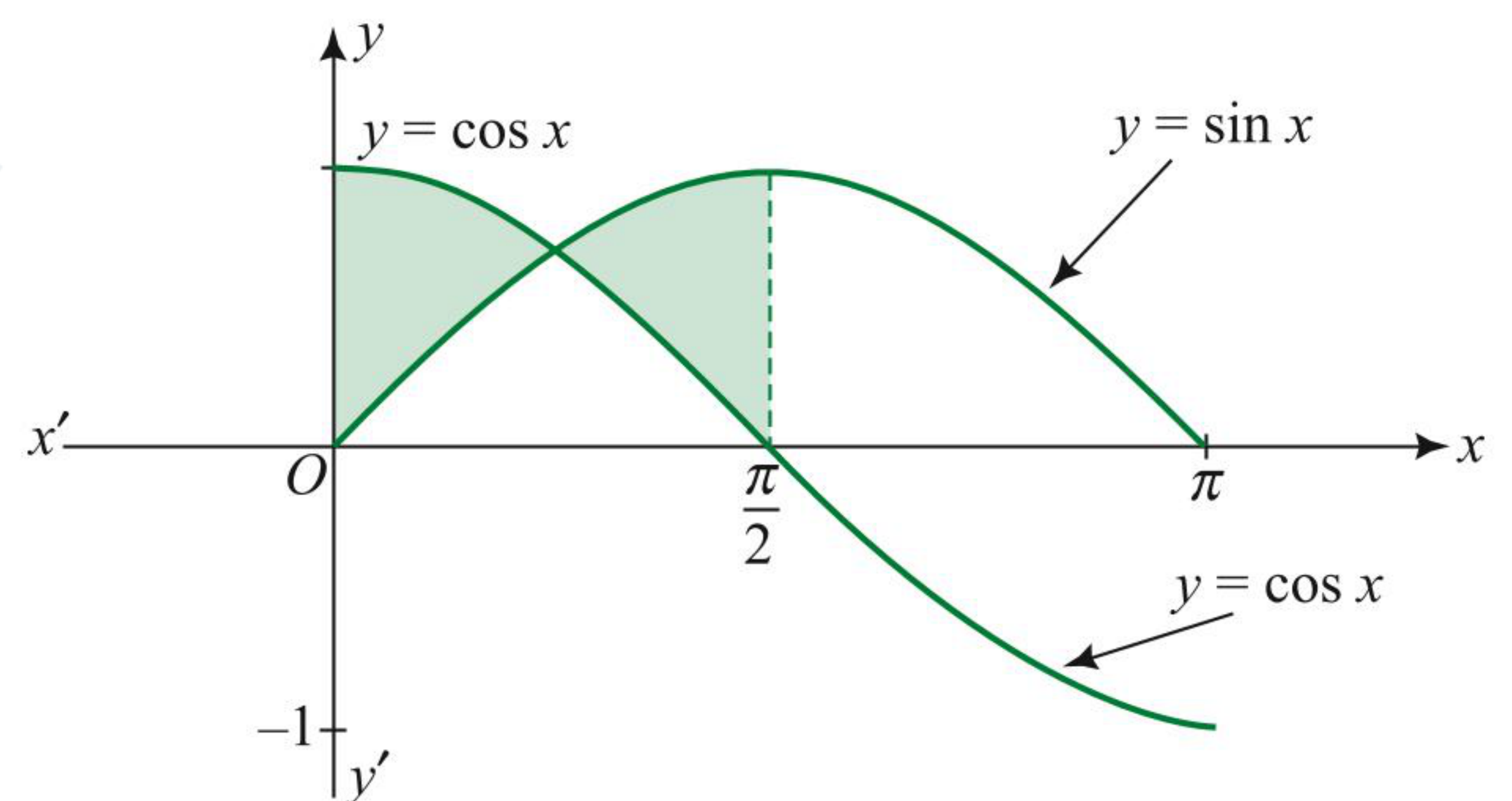
$$\begin{aligned} &= \text{Area of semicircle} + 2 \int_0^2 (\sqrt{16 - x^2} - \sqrt{6x}) dx \\ &= 8\pi + 2 \left[\frac{x}{2} \sqrt{16 - x^2} + \frac{16}{2} \sin^{-1} \frac{x}{4} - \sqrt{6} \frac{2x^{3/2}}{3} \right]_0^2 \\ &= 8\pi + 2 \left[2\sqrt{3} + \frac{4\pi}{3} - \sqrt{6} \frac{2 \cdot 2^{3/2}}{3} \right] \\ &= \frac{4}{3} (8\pi - \sqrt{3}) \text{ sq. units} \end{aligned}$$

6. $y = x^2 + 2$ is parabola having vertex at $(0, 2)$ and having concavity upward.



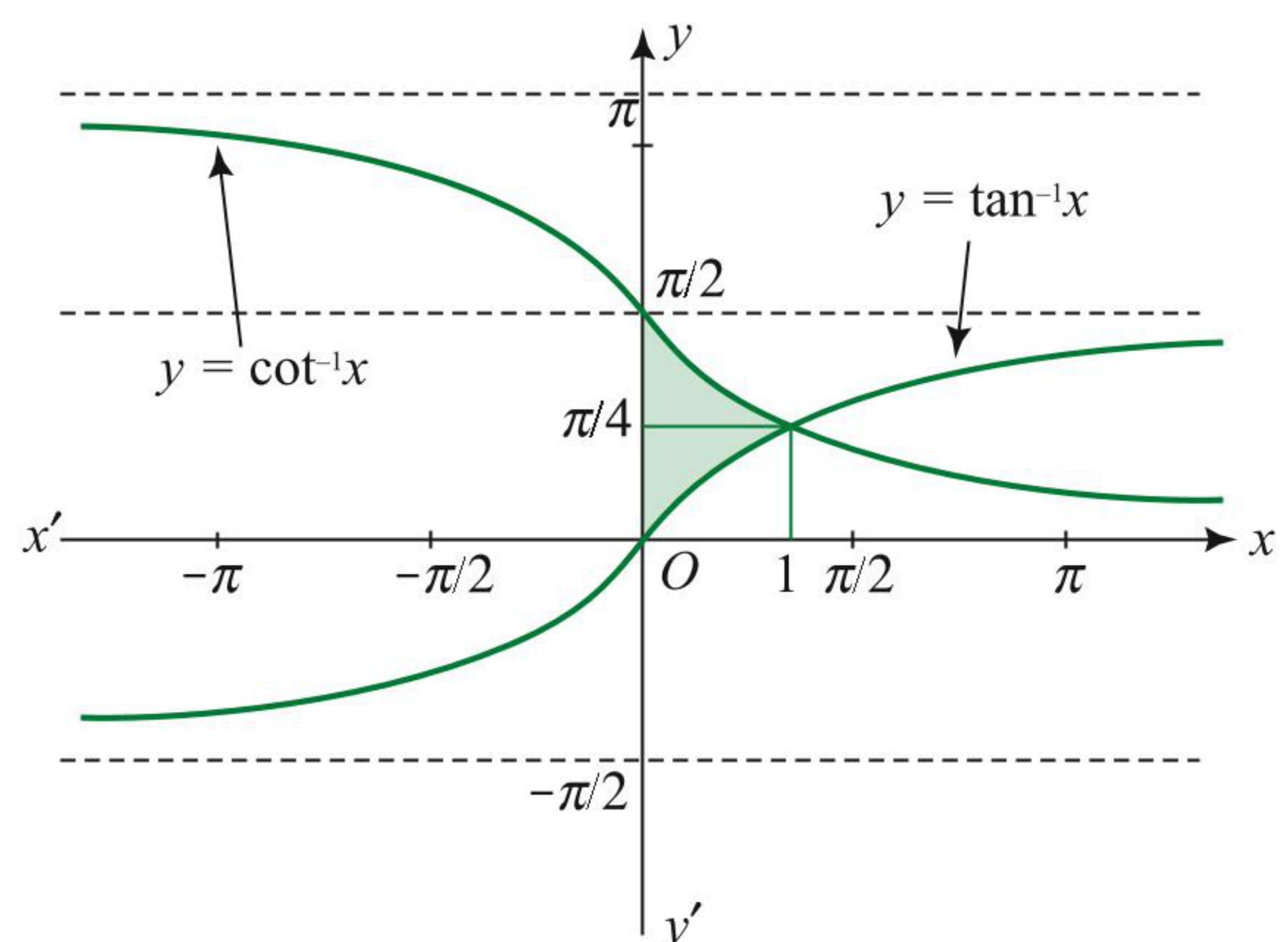
$$\begin{aligned} \text{Then, required area} &= \int_0^3 ((x^2 + 2) - x) dx = \left[\frac{x^3}{3} + 2x \right]_0^3 - \left[\frac{x^2}{2} \right]_0^3 \\ &= [9 + 6] - \left[\frac{9}{2} \right] = \frac{21}{2} \text{ sq. units.} \end{aligned}$$

7.



$$\begin{aligned} \text{Required area} &= \int_0^{\pi/2} |\sin x - \cos x| dx \\ &= 2 \int_0^{\pi/4} (\cos x - \sin x) dx \\ &= 2 [\sin x + \cos x]_0^{\pi/4} \\ &= 2 \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right) = 2(\sqrt{2} - 1) \text{ sq. units} \end{aligned}$$

8.



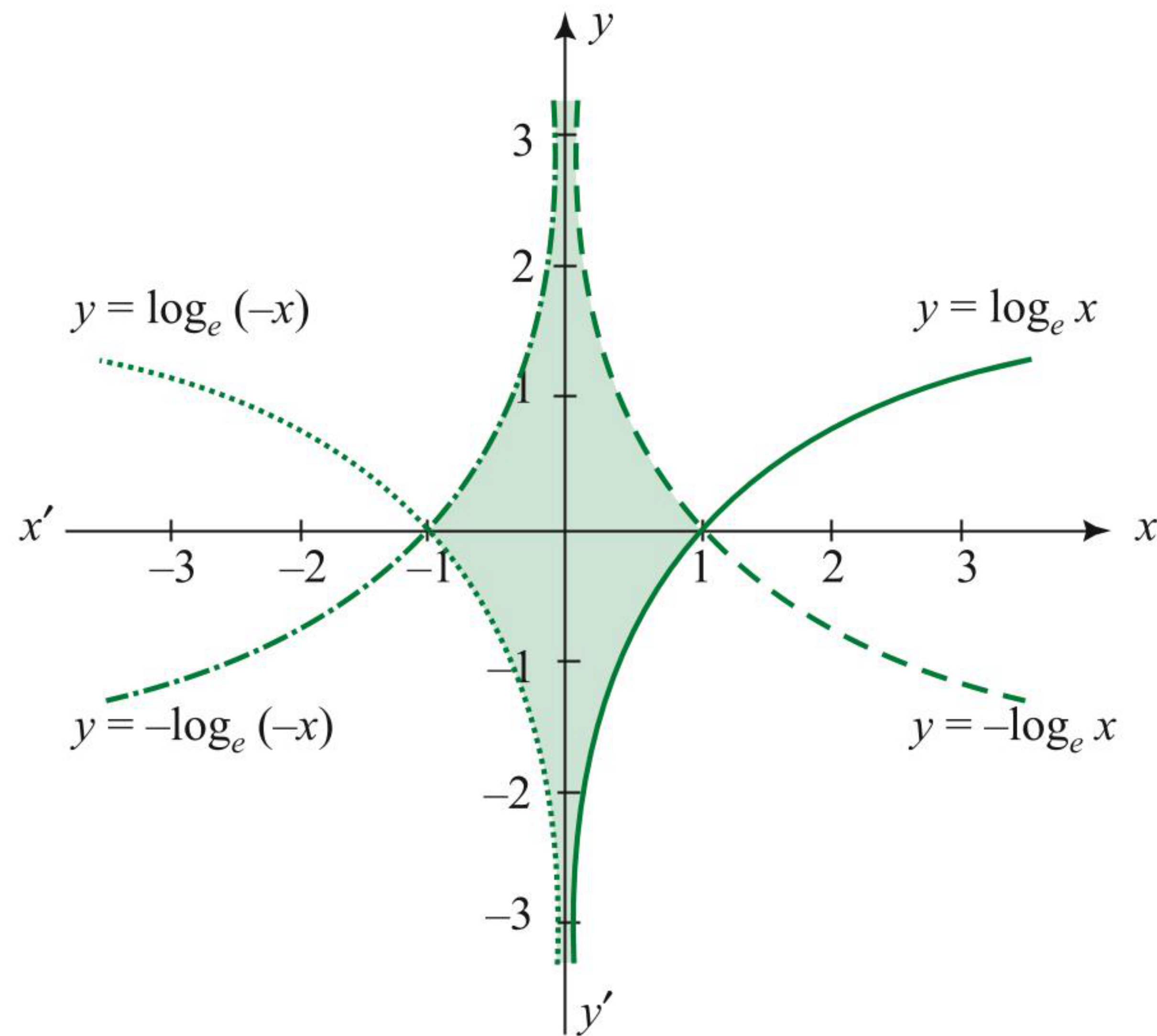
Integrating along x -axis, we get

$$A = \int_0^1 (\cot^{-1} x - \tan^{-1} x) dx = \int_0^1 \left(\frac{\pi}{2} - 2 \tan^{-1} x \right) dx$$

Integrating along y -axis, we get

$$A = 2 \int_0^{\pi/4} x dy = 2 \int_0^{\pi/4} \tan y dy = [\log(\sec y)]_0^{\pi/4} = \log \sqrt{2} \text{ sq. units}$$

9.



From the figure, required area = area of shaded region
 $= 1 + 1 + 1 + 1 = 4$ sq. units.

10. Given inequalities are

$$y^2 \leq 4x \quad (1)$$

$$\text{and } 4x^2 + 4y^2 \leq 9 \quad (2)$$

Points satisfying (1) lies on or in interior to parabola $y^2 = 4x$ and those of (2) lies on or inside circle $4x^2 + 4y^2 = 9$

Solving we get,

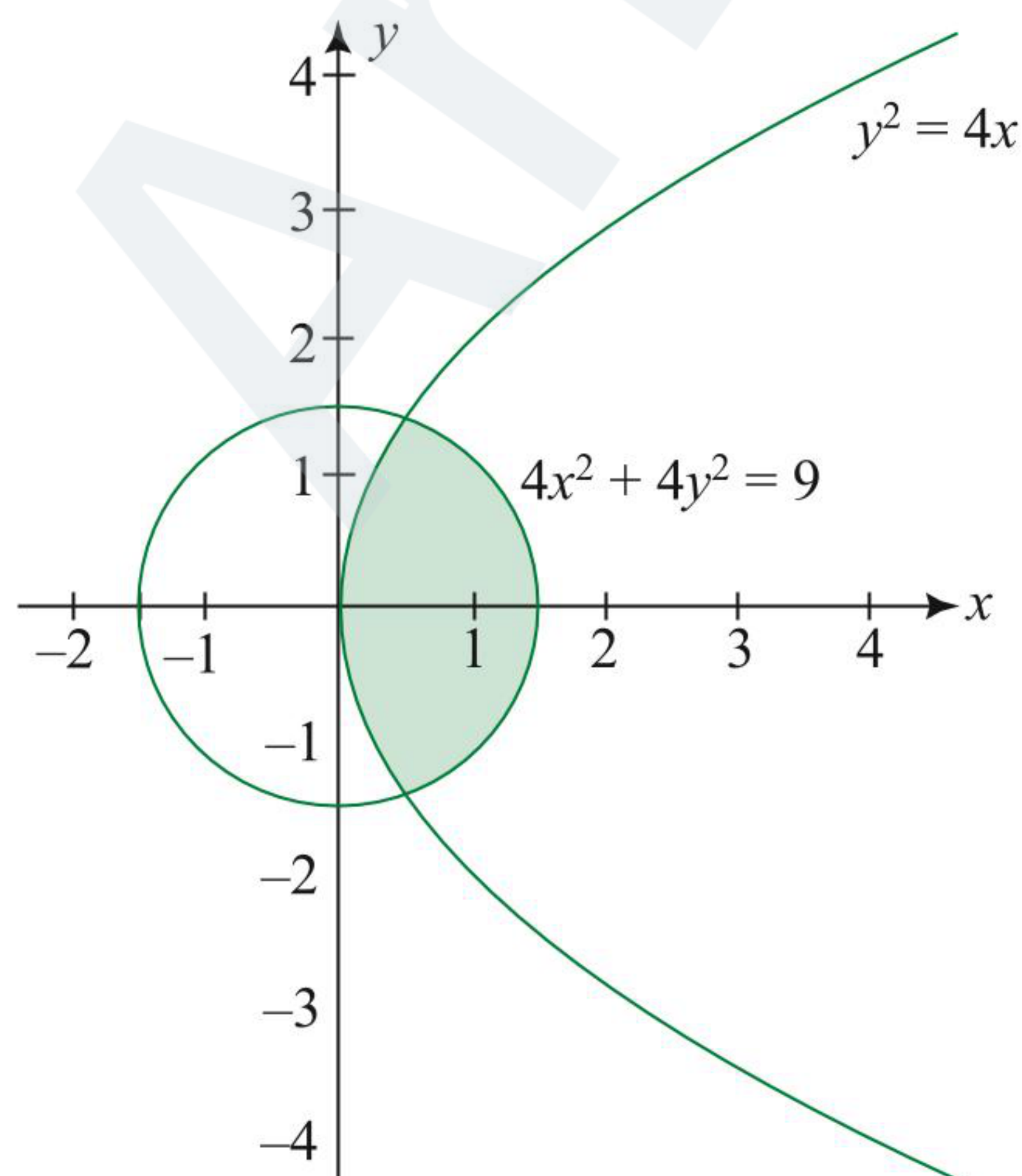
$$4x^2 + 16x = 9$$

$$\text{or } (2x - 1)(2x + 9) = 0$$

$$\text{or } x = 1/2 \quad (\text{as } x = -9 \text{ not possible})$$

Therefore, the points of intersection of both curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

The graph of these two curves and the common region of the points satisfying both the inequalities is as shown in the figure.



From the figure, required area is given by

$$A = 2 \int_1^{\frac{1}{2}} 2\sqrt{x} dx + 2 \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2} \sqrt{9 - 4x^2} dx$$

Putting $2x = t$ in the second integral, we get

$$\begin{aligned} dx &= \frac{dt}{2} \\ \therefore A &= 2 \left[\int_0^{\frac{1}{2}} 2\sqrt{x} dx + \frac{1}{4} \int_1^3 \sqrt{(3)^2 - (t)^2} dt \right] \\ &= 2 \left[\left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^{\frac{1}{2}} + \frac{1}{4} \left[\frac{t}{2} \sqrt{9 - t^2} + \frac{9}{2} \sin^{-1} \left(\frac{t}{3} \right) \right]_1^3 \right] \\ &= 2 \left[\frac{2}{3\sqrt{2}} + \frac{1}{4} \left[\left\{ 0 + \frac{9}{2} \sin^{-1}(1) \right\} - \left\{ \frac{1}{2} \sqrt{8} + \frac{9}{2} \sin^{-1} \left(\frac{1}{3} \right) \right\} \right] \right] \\ &= 2 \left(\frac{9\pi}{16} - \frac{9}{8} \sin^{-1} \left(\frac{1}{3} \right) + \frac{\sqrt{2}}{12} \right) \end{aligned}$$

11. The given curves are

$$y = \sqrt{5 - x^2} \quad (1)$$

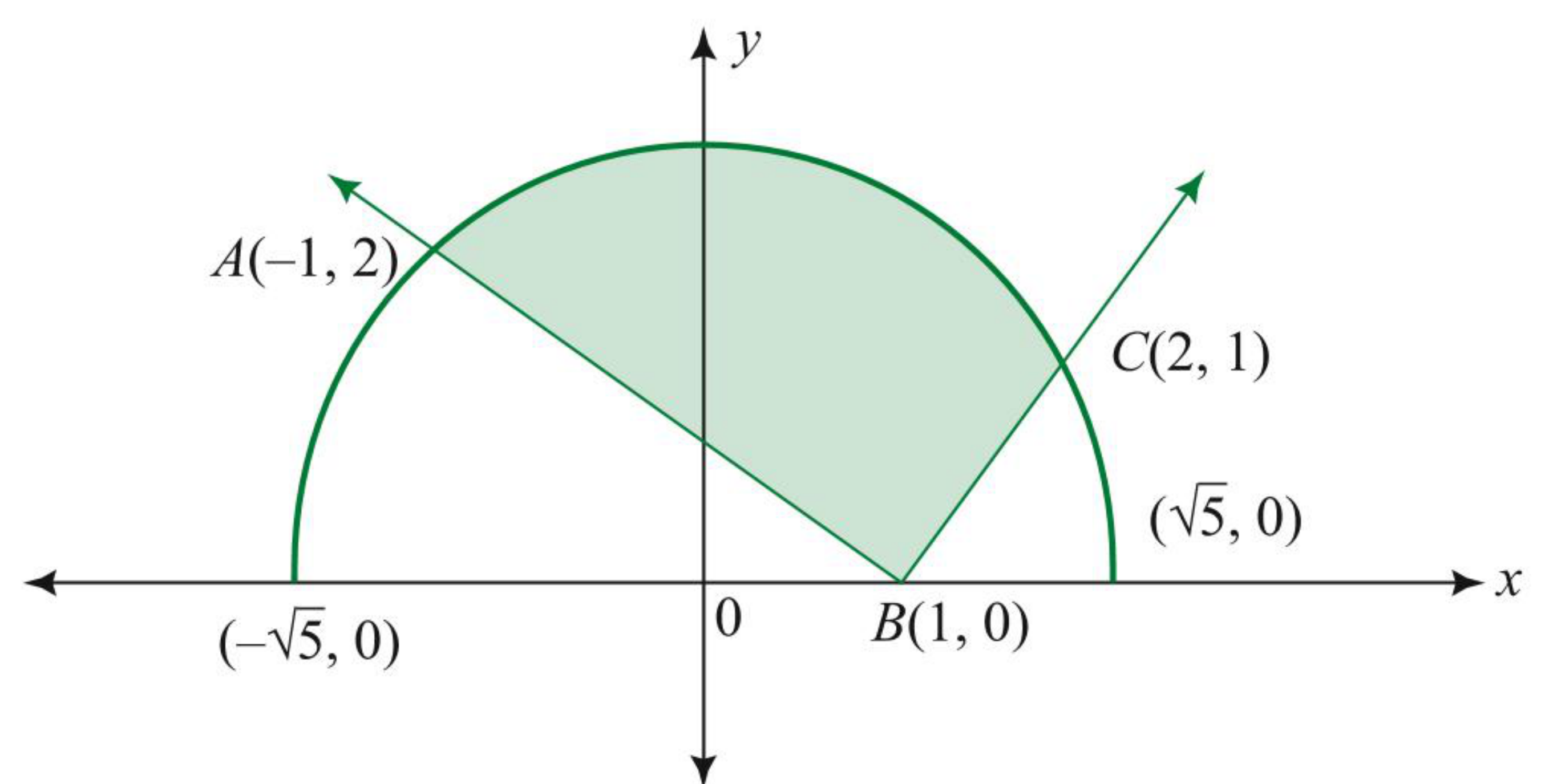
$$y = |x - 1| \quad (2)$$

We can clearly see that on squaring the both sides of (1), equation (2) represents a circle.

But as y is +ve square root, (1) represents the upper-half of the circle with centre $(0, 0)$ and radius $\sqrt{5}$.

Equation (2) represents the curve

$$y = \begin{cases} -x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$$



Graph of these curves are as shown in the figure with point of intersection of $y = \sqrt{5 - x^2}$ and $y = -x + 1$ as $A(-1, 2)$

and of $y = \sqrt{5 - x^2}$ and $y = x - 1$ as $C(2, 1)$

$\therefore$ Required area = Shaded area

$$\begin{aligned} &= \int_{-1}^2 \sqrt{5 - x^2} dx - \int_{-1}^2 |x - 1| dx \\ &= \left[\frac{x}{2} \sqrt{5 - x^2} + \frac{5}{2} \sin^{-1} \left(\frac{x}{\sqrt{5}} \right) \right]_{-1}^2 - \int_{-1}^1 -(x - 1) dx - \int_1^2 (x - 1) dx \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{2}{2} \sqrt{5-4} + \frac{5}{2} \sin^{-1} \frac{2}{\sqrt{5}} \right) - \left(\frac{-1}{2} \sqrt{5-1} \right. \\
&\quad \left. + \frac{5}{2} \sin^{-1} \left(\frac{-1}{\sqrt{5}} \right) \right) - \left(\frac{-x^2}{2} + x \right)_{-1}^1 - \left(\frac{x^2}{2} - x \right)_1^2 \\
&= 1 + \frac{5}{2} \sin^{-1} \frac{2}{\sqrt{5}} + 1 + \frac{5}{2} \sin^{-1} \left(\frac{1}{\sqrt{5}} \right) \\
&\quad - \left[\left(\frac{-1}{2} + 1 \right) - \left(\frac{-1}{2} - 1 \right) \right] - \left[(2-2) - \left(\frac{1}{2} - 1 \right) \right] \\
&= 2 + \frac{5}{2} \left[\sin^{-1} \frac{2}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{5}} \right] - 2 - \frac{1}{2} \\
&= \frac{5}{2} \left[\sin^{-1} \frac{2}{\sqrt{5}} + \cos^{-1} \frac{2}{\sqrt{5}} \right] - \frac{1}{2} = \frac{5}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2} \\
&= \frac{5\pi - 2}{4} \text{ sq. units.}
\end{aligned}$$

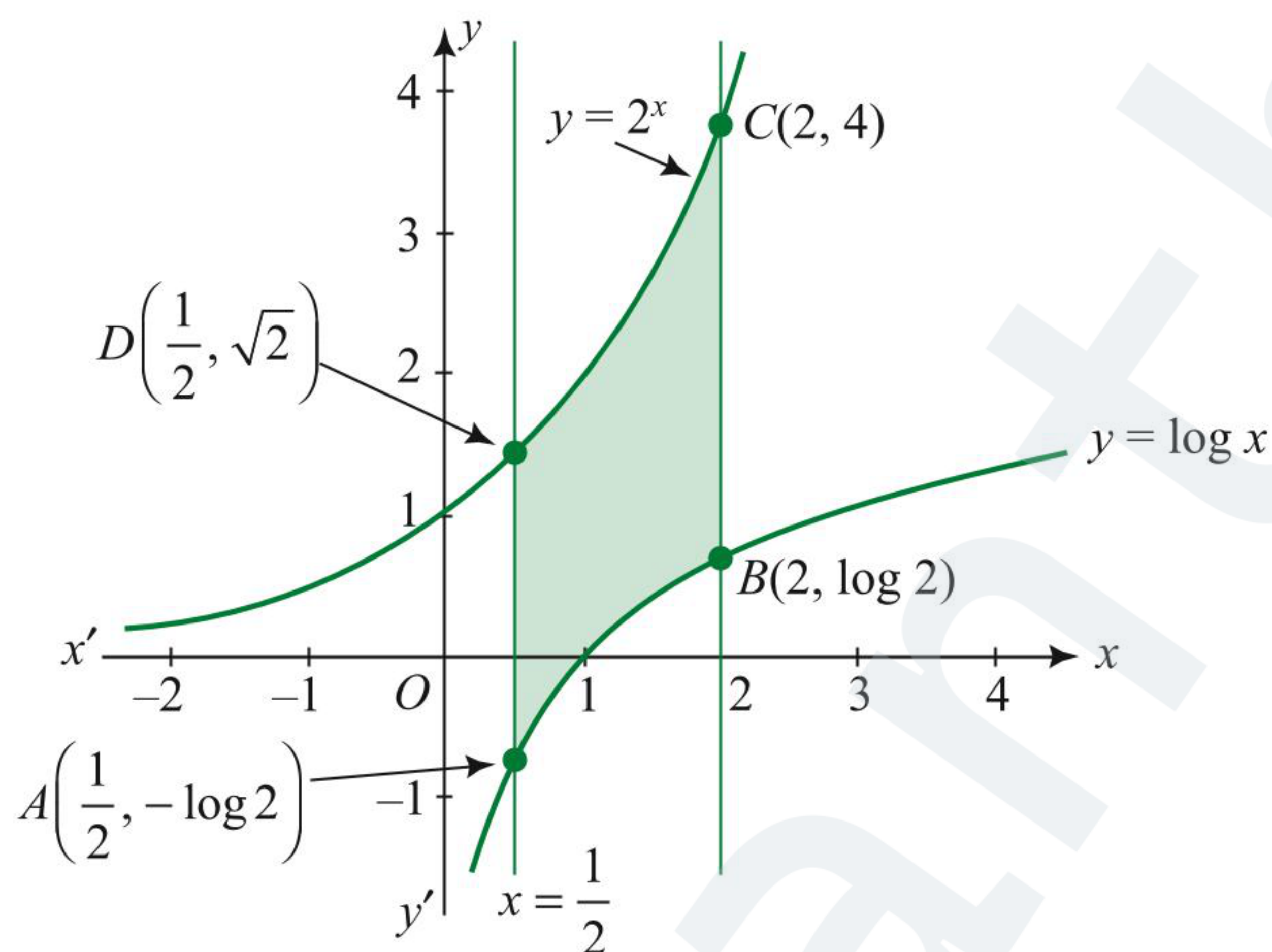
12. The given curves are

$$x = \frac{1}{2} \quad (1)$$

$$x = 2 \quad (2)$$

$$y = \log_e x \quad (3)$$

$$y = 2^x \quad (4)$$



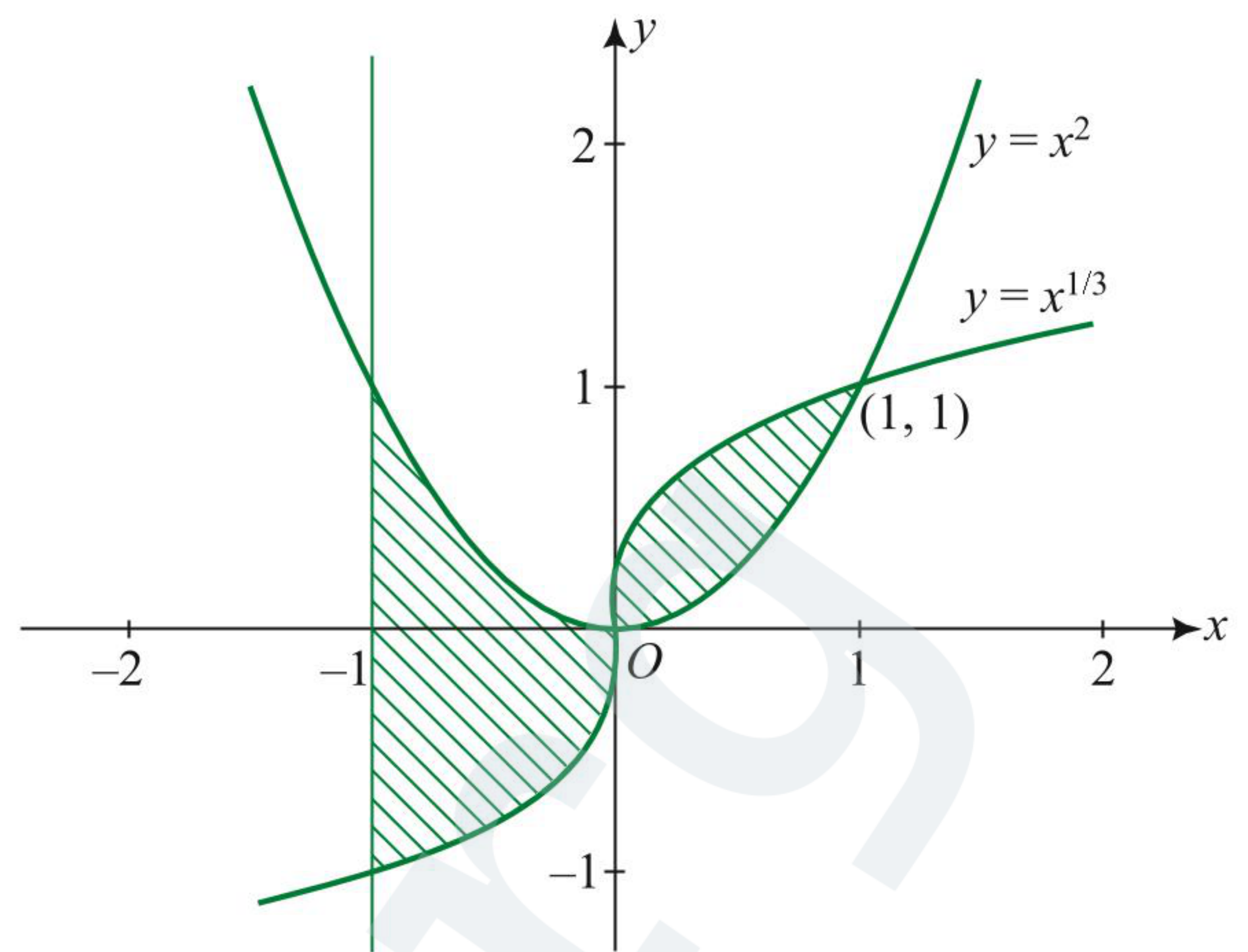
Required area = Area of ABCDA

$$\begin{aligned}
&= \int_{1/2}^2 (2^x - \log x) dx \\
&= \left[\frac{2^x}{\log 2} - (x \log x - x) \right]_{1/2}^2 \\
&= \left(\frac{4}{\log 2} - 2 \log 2 + 2 \right) - \left(\frac{\sqrt{2}}{\log 2} - \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \right) \\
&= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2} \log 2 + \frac{3}{2} \right) \text{ sq. units}
\end{aligned}$$

13. Solving given curves we get $x^2 = x^{1/3}$.

$$\therefore x = 0 \text{ or } x = 1$$

$$\text{Required area, } A = \int_{-1}^0 [x^2 - x^{1/3}] dx + \int_0^1 [x^{1/3} - x^2] dx$$



$$\begin{aligned}
&= \left[\frac{x^3}{3} - \frac{3x^{4/3}}{4} \right]_{-1}^0 + \left[\frac{3x^{4/3}}{4} - \frac{x^3}{3} \right]_0^1 \\
&= - \left[-\frac{1}{3} - \frac{3}{4} \right] + \left[\frac{3}{4} - \frac{1}{3} \right] = \frac{3}{2} \text{ sq. units.}
\end{aligned}$$

14. Solving given curves, we have $x - \sin x = x + \cos x$

$$\therefore \tan x = -1$$

For smallest bounded area we must consider two consecutive solutions,

$$\text{i.e., } x = -\frac{\pi}{4}, \frac{3\pi}{4}$$

$$\begin{aligned}
\therefore \text{ Required area} &= \left| \int_{-\pi/4}^{3\pi/4} [(x + \cos x) - (x - \sin x)] dx \right| \\
&= \left| \int_{-\pi/4}^{3\pi/4} [\cos x + \sin x] dx \right| \\
&= [\sin x - \cos x]_{-\pi/4}^{3\pi/4} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right)
\end{aligned}$$

Exercise 9.3

1. According to the given conditions

$$\int_0^t [f(x) - (x^4 - 4x^2)] dx = k \int_0^t [(2x^2 - x^3) - f(x)] dx$$

Differentiating both sides w.r.t. t , we get

$$f(t) - (t^4 - 4t^2) = k(2t^2 - t^3 - f(t))$$

$$\text{or } (1+k)f(t) = k2t^2 - kt^3 + t^4 - 4t^2$$

$$\Rightarrow f(t) = \frac{1}{k+1} [t^4 - kt^3 + (2k-4)t^2]$$

Hence, required f is given by $f(x) = \frac{1}{k+1} (x^4 - kx^3 + 2(k-2)x^2)$.

$$2. \text{ Area} = \int_1^b f(x) dx = \sqrt{b^2 + 1} - \sqrt{2}$$

Differentiating both sides w.r.t. b , we get

$$f(b) = \frac{b}{\sqrt{b^2 + 1}}$$

$$\therefore f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

3. According to the question $\int_0^a f(x) dx = \frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$

$$\therefore f(a) = \frac{d}{da} \left(\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a \right)$$

$$= a + \frac{a}{2} \cos a + \frac{1}{2} \sin a - \frac{\pi}{2} \sin a$$

$$\therefore f\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

4. According to the question $\int_0^x f(x) dx = \lambda \{f(x)\}^{n+1}$

Where λ is constant of proportionality

Differentiating both sides w.r.t x ,

$$f(x) = \lambda(n+1)(f(x))^n f'(x)$$

or $(f(x)^{n-1}) f'(x) = \frac{1}{\lambda(n+1)}$

Integrating both sides w.r.t x , $\frac{(f(x))^n}{n} = \frac{x}{\lambda(n+1)} + C$

$$f(0) = 0, \therefore C = 0$$

$$(f(x))^n = \frac{nx}{\lambda(n+1)}$$

$$f(1) = 1$$

$$\therefore \frac{n}{\lambda(n+1)} = 1$$

$$\therefore f(x) = x^{1/n}$$

5. $R_1 : |x+y| + |x-y| \leq 4$

Considering $x+y+x-y \leq 4$, we get $x \leq 2$

Similarly, for other cases, we get $x \geq -2$, $y \leq 2$ and $y \geq -2$

$\therefore R_1$ is the square bounded by lines $x = \pm 2$ and $y = \pm 2$

$$R_2 : |x| \leq 1$$

$$\therefore -1 \leq x \leq 1$$

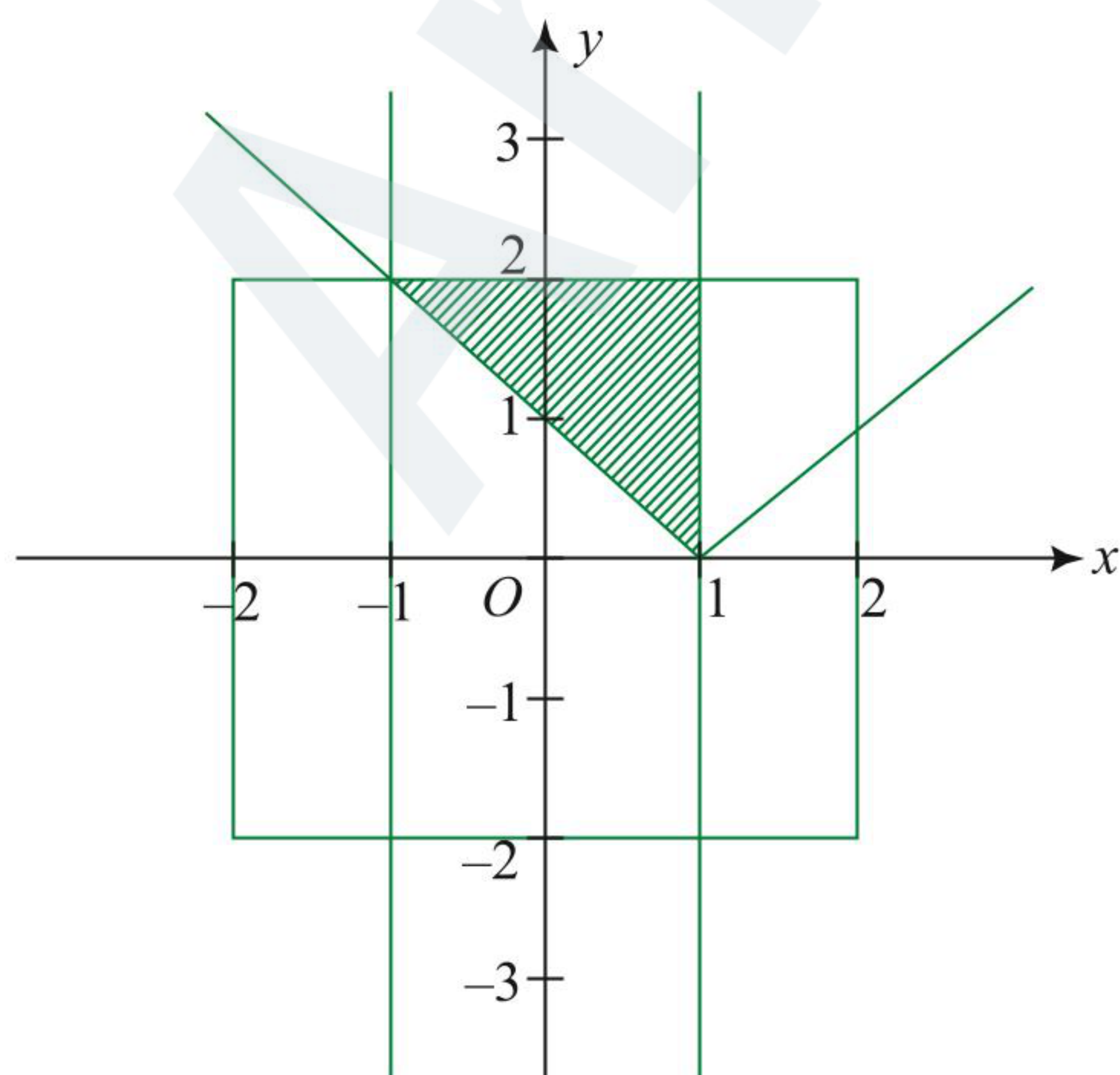
$\therefore R_2$ is the region of all points between $x = \pm 1$

$$y \geq \sqrt{x^2 - 2x + 1}$$

$$R_3 : y \geq |x-1|$$

$\therefore R_3$ is the open triangular region above x -axis between the lines $y = 1-x$ and $y = x-1$

$\therefore$ Required common region is as shown in the following figure.



Required Area = Area of the shaded portion

$$= \frac{1}{2} \times 2 \times 2 \text{ sq. units} = 2 \text{ sq. units}$$

6. R_1 : points $P(x, y)$ are nearer to $(1, 0)$ than to $x = -1$

$$\therefore \sqrt{(x-1)^2 + y^2} < |x+1|$$

$$\Rightarrow y^2 < 4x$$

$$\Rightarrow \text{point } P \text{ lie inside parabola } y^2 = 4x$$

R_2 : Points $P(x, y)$ are nearer to $(0, 0)$ than to $(8, 0)$

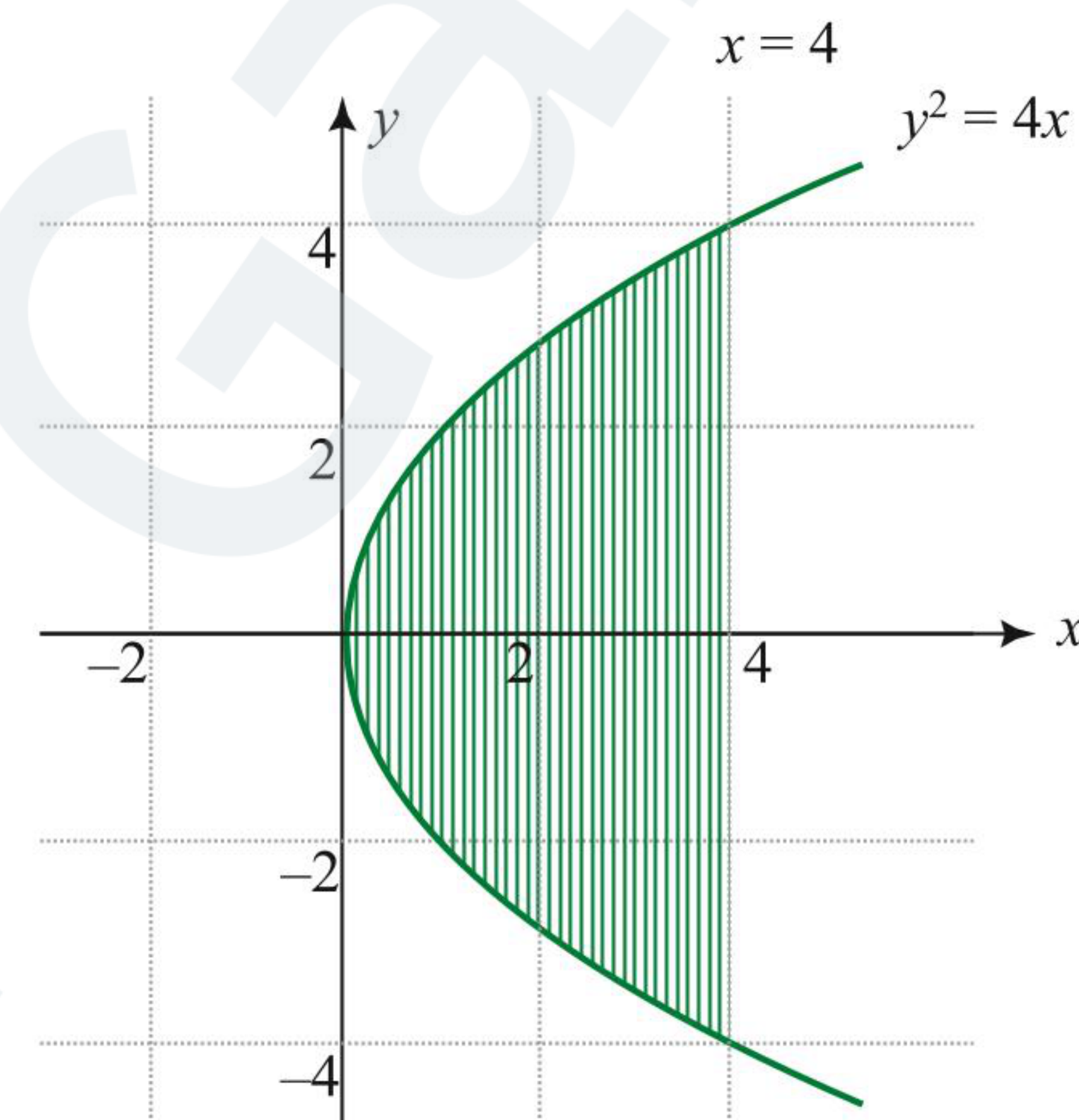
$$\therefore |x| < |x-8|$$

$$\Rightarrow x^2 < x^2 - 16x + 64$$

$$\Rightarrow x < 4$$

$$\Rightarrow \text{point } P \text{ lie to the left side of line } x = 4$$

The area of common region of R_1 and R_2 is area bounded by $x = 4$ and $y^2 = 4x$.



$$\therefore \text{Required area} = 2 \int_0^4 \left(4 - \frac{y^2}{4} \right) dy \quad (\text{Integrating along } x\text{-axis})$$

$$= 2 \left[4y - \frac{y^3}{12} \right]_0^4$$

$$= 2 \left[16 - \frac{64}{12} \right]$$

$$= \frac{64}{3} \text{ sq. units}$$

7. Required area, $A = 2 \int_0^{1/2} f^{-1}(x) dx$

Let $f^{-1}(x) = t$

$$\Rightarrow A = 2 \int_0^1 t f'(t) dt$$

$$= 2 [t f(t)]_0^1 - 2 \int_0^1 f(t) dt$$

$$= 2f(1) - \int_0^1 \frac{2t}{1+t^2} dt$$

$$= 1 - [\log(1+t^2)]_0^1$$

$$= 1 - \log 2$$

Exercises

Single Correct Answer Type

1. (1) Clearly
- t
- can be any real number

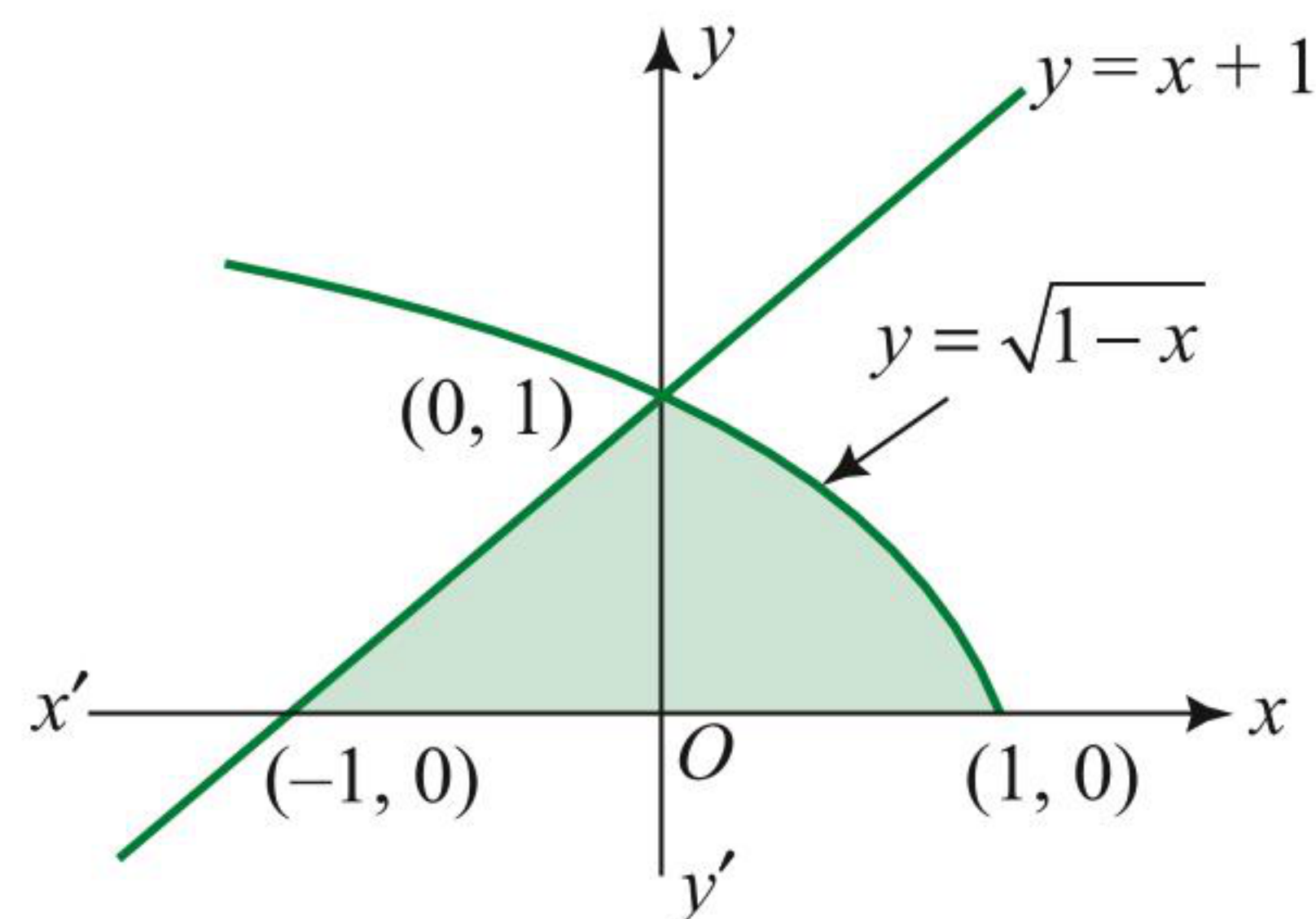
$$\text{Let } t = \tan \theta \Rightarrow x = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\Rightarrow x = \cos 2\theta \text{ and } y = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \sin 2\theta$$

$$\Rightarrow x^2 + y^2 = 1$$

Thus, required area is π sq. units.

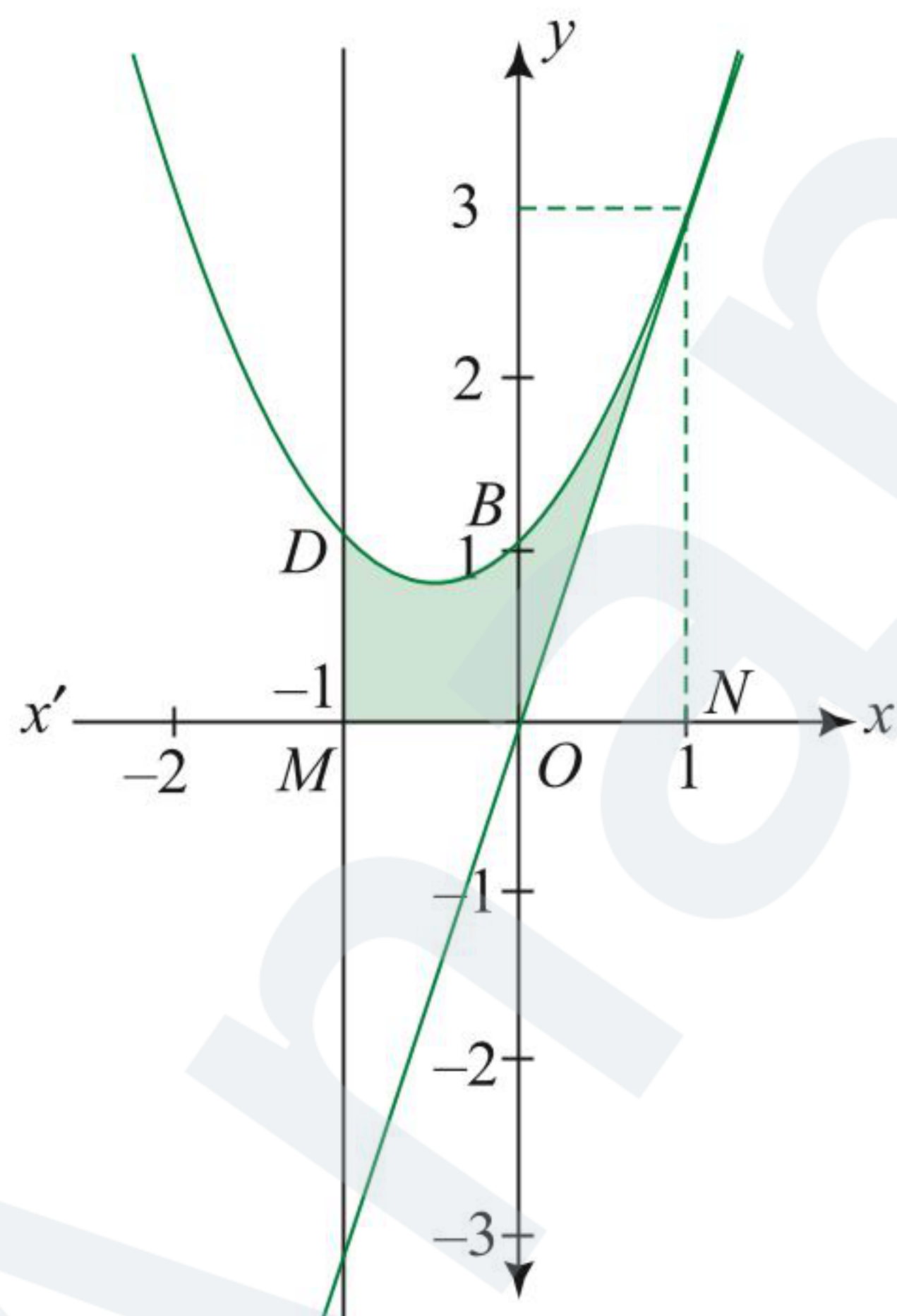
2. (4)



Required area = Shaded region

$$\begin{aligned} &= \int_0^1 (x_2 - x_1) dy \text{ (integrating along } y\text{-axis)} \\ &= \int_0^1 [(1 - y^2) - (y - 1)] dy \\ &= \frac{7}{6} \text{ sq. unit} \end{aligned}$$

3. (3) Given
- $y = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \Rightarrow y - \frac{3}{4} = \left(x + \frac{1}{2}\right)^2$
- .



This is a parabola with vertex at $\left(-\frac{1}{2}, \frac{3}{4}\right)$ and the curve is concave upwards.

$$y = x^2 + x + 1 \text{ or } \frac{dy}{dx} = 2x + 1 \text{ or } \left(\frac{dy}{dx}\right)_{(1,3)} = 3$$

Equation of the tangent at $A(1, 3)$ is $y = 3x$

Required (shaded) area = Area $ABDMN$ - Area ONA

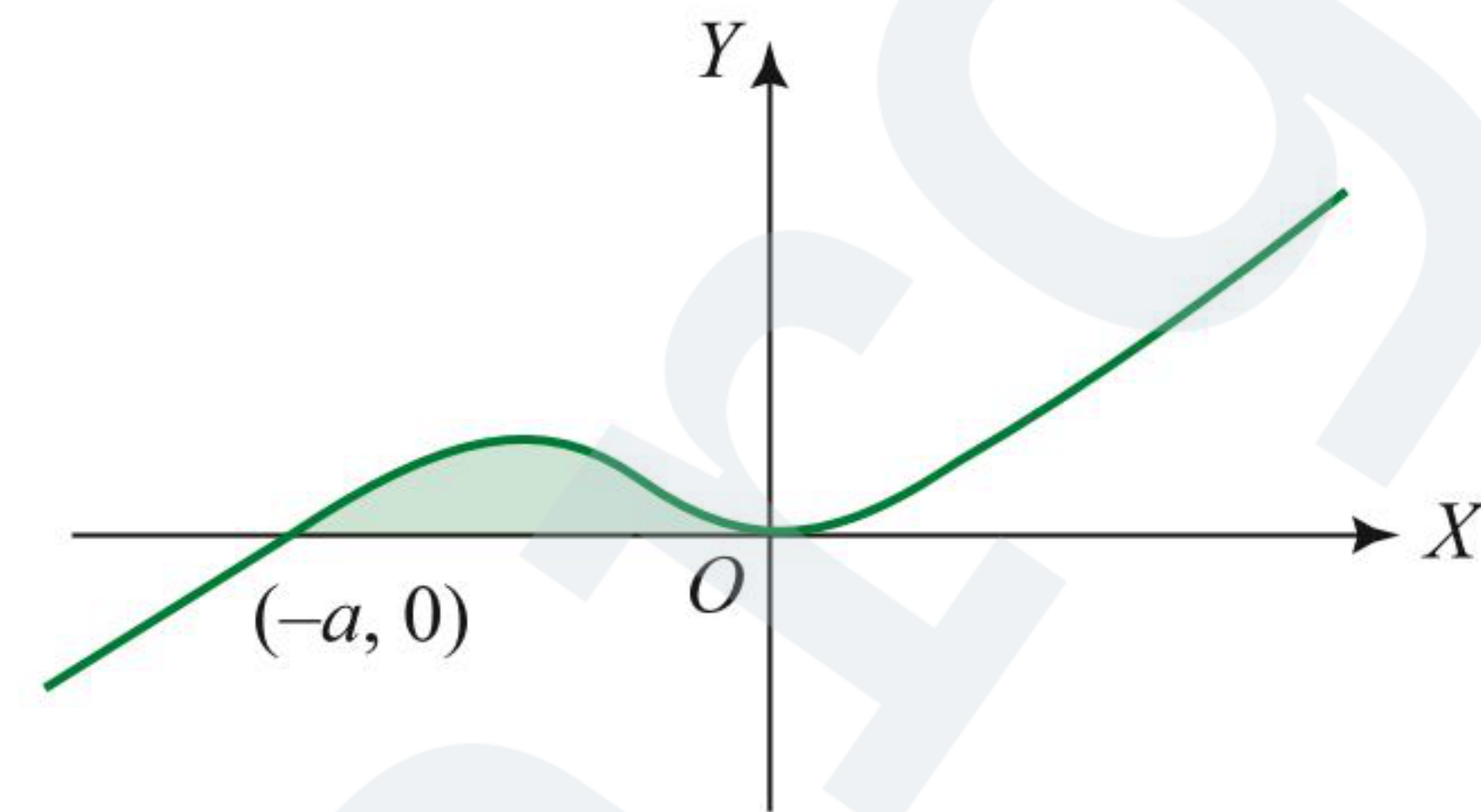
$$\text{Now, area } ABDMN = \int_{-1}^1 (x^2 + x + 1) dx = 2 \int_0^1 (x^2 + 1) dx = \frac{8}{3}$$

$$\text{Area of } ONA = \frac{1}{2} \times 1 \times 3 = \frac{3}{2}$$

$$\therefore \text{ Required area} = \frac{8}{3} - \frac{3}{2} = \frac{16-9}{6} = \frac{7}{6} \text{ sq. units.}$$

4. (4) The curve is
- $y = \frac{x^2(x+a)}{a^2}$
- , which is a cubic polynomial.

Since $\frac{x^2(x+a)}{a^2} = 0$ has a repeated root $x = 0$, it touches x -axis at $(0, 0)$ and intersects at $(-a, 0)$.



$$\begin{aligned} \text{Required area} &= \int_{-a}^0 y dx \\ &= \int_{-a}^0 \left[\frac{x^2(x+a)}{a^2} \right] dx \\ &= a^2/12 \text{ sq. units.} \end{aligned}$$

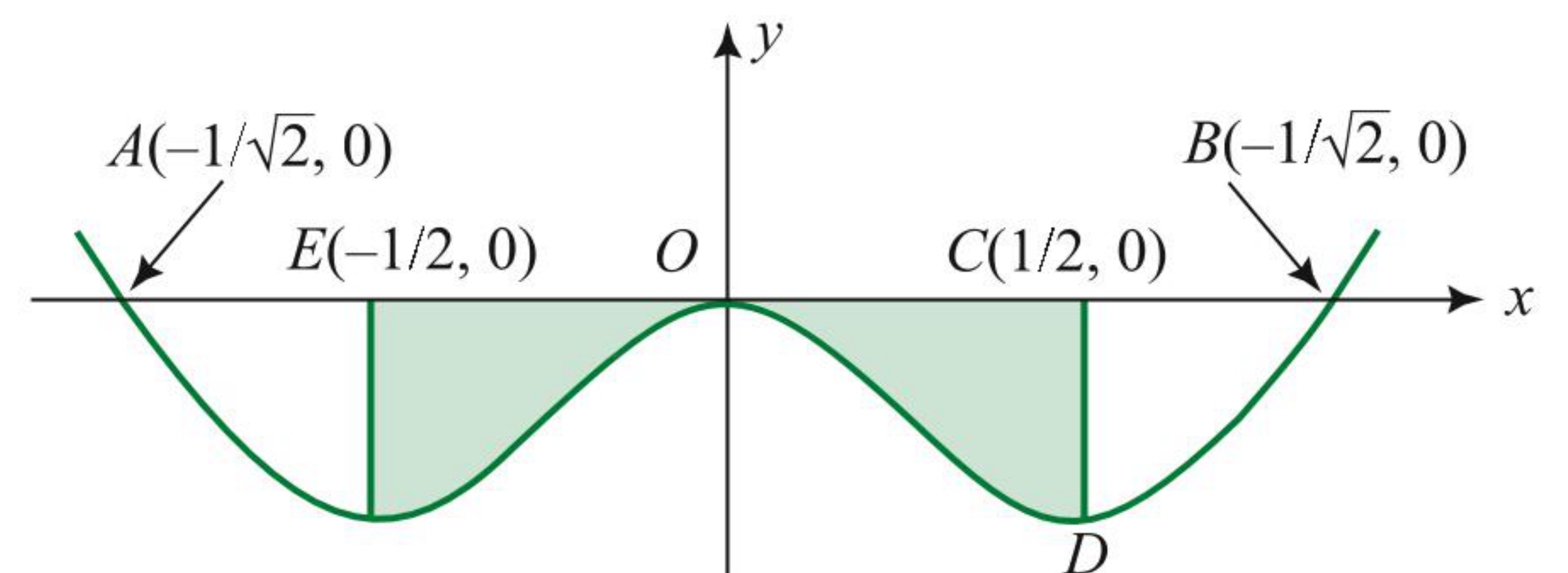
5. (2) The curve is
- $y = 2x^4 - x^2 = x^2(2x^2 - 1)$

The curve is symmetrical about the axis of y .

Also, it is a polynomial of 4 degree having roots $0, 0, \pm \frac{1}{\sqrt{2}}$. $x = 0$ is a repeated root. Hence, graph touches at $(0, 0)$.

The curve intersects the axes at $O(0, 0)$, $A(-1/\sqrt{2}, 0)$ and $B(1/\sqrt{2}, 0)$.

Thus, the graph of the curve is as shown in the figure.

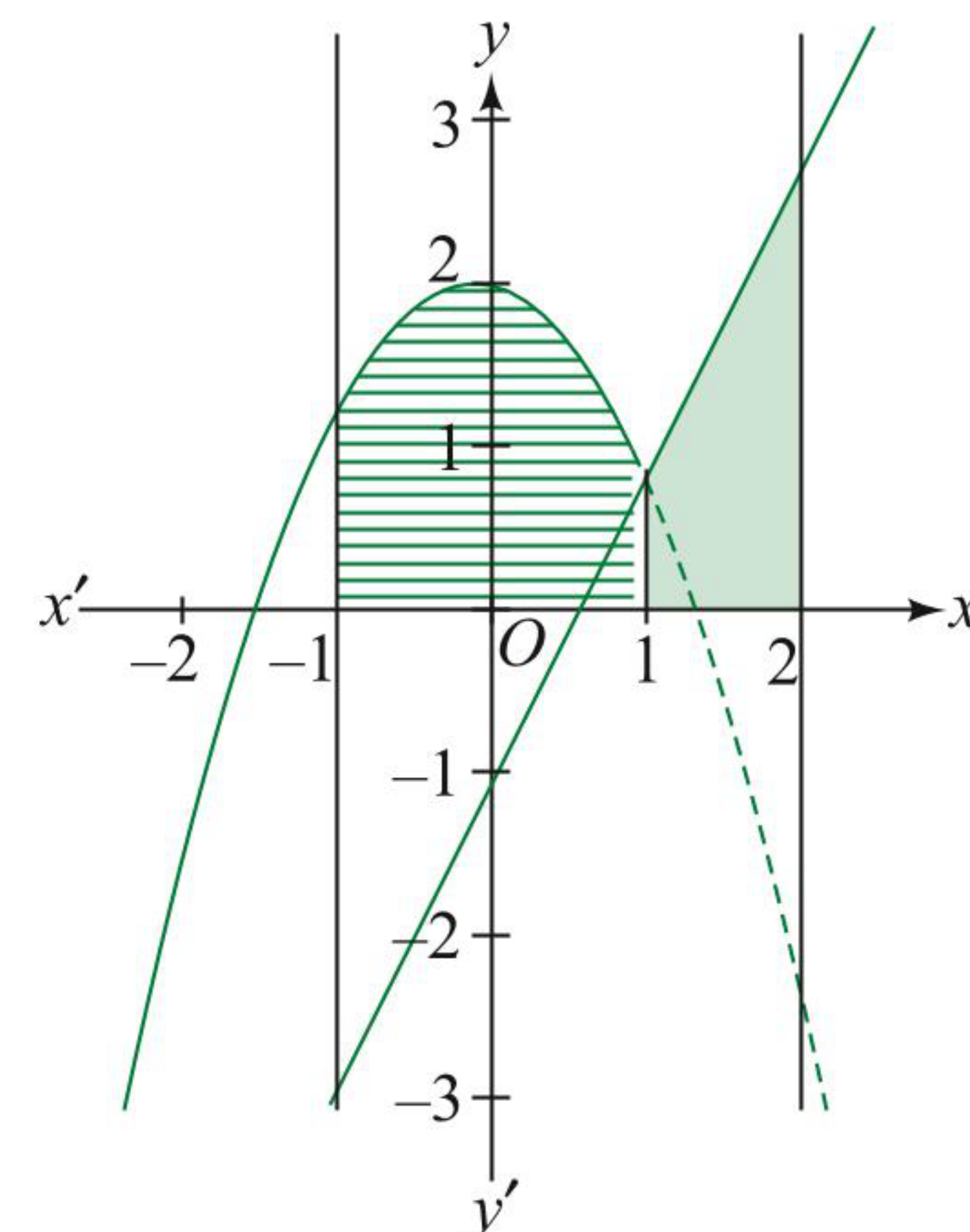


Here, $y \leq 0$, as x varies from $x = -1/2$ to $x = 1/2$

$\therefore$ required area = 2 Area $OCDO$

$$\begin{aligned} &= 2 \left| \int_0^{1/2} y dx \right| \\ &= 2 \left| \int_0^{1/2} (2x^4 - x^2) dx \right| \\ &= 7/120 \text{ sq. units} \end{aligned}$$

6. (1)



$$\begin{aligned}
 A &= \int_{-1}^1 (-x^2 + 2)dx + \int_1^2 (2x - 1)dx \\
 &= \left(-\frac{x^3}{3} + 2x \right)_{-1}^1 + (x^2 - x)_1^2 \\
 &= \frac{16}{3} \text{ sq. units}
 \end{aligned}$$

7. (3) $a^2x^2 + ax + 1$ is clearly positive for all real values of x . Area under consideration

$$\begin{aligned}
 A &= \int_0^1 (a^2x^2 + ax + 1)dx \\
 &= \frac{a^2}{3} + \frac{a}{2} + 1 \\
 &= \frac{1}{6} (2a^2 + 3a + 6) \\
 &= \frac{1}{6} \left(2 \left(a^2 + \frac{3}{2}a + \frac{9}{16} \right) + 6 - \frac{18}{16} \right) \\
 &= \frac{1}{6} \left(2 \left(a + \frac{3}{4} \right)^2 + \frac{39}{8} \right),
 \end{aligned}$$

which is clearly minimum for $a = -\frac{3}{4}$.

$$\begin{aligned}
 8. (3) \quad A &= \int_{\frac{2\pi}{3k}}^{\frac{\pi}{k}} (\sin kx)dx - \int_{\frac{\pi}{k}}^{\frac{5\pi}{3k}} (\sin kx)dx \\
 &= -\left(\frac{\cos kx}{k} \right)_{\frac{2\pi}{3k}}^{\frac{\pi}{k}} + \left(\frac{\cos kx}{k} \right)_{\frac{\pi}{k}}^{\frac{5\pi}{3k}} \\
 &= -\frac{1}{k} \left[-1 - \cos \frac{2\pi}{3} \right] + \frac{1}{k} \left[\cos \frac{5\pi}{3} - \cos \pi \right] \\
 &= -\frac{1}{k} \left[-1 + \frac{1}{2} \right] + \frac{1}{k} \left[\frac{1}{2} + 1 \right] \\
 &= \frac{1}{2k} + \frac{3}{2k} \\
 &= \frac{2}{k} < 2
 \end{aligned}$$

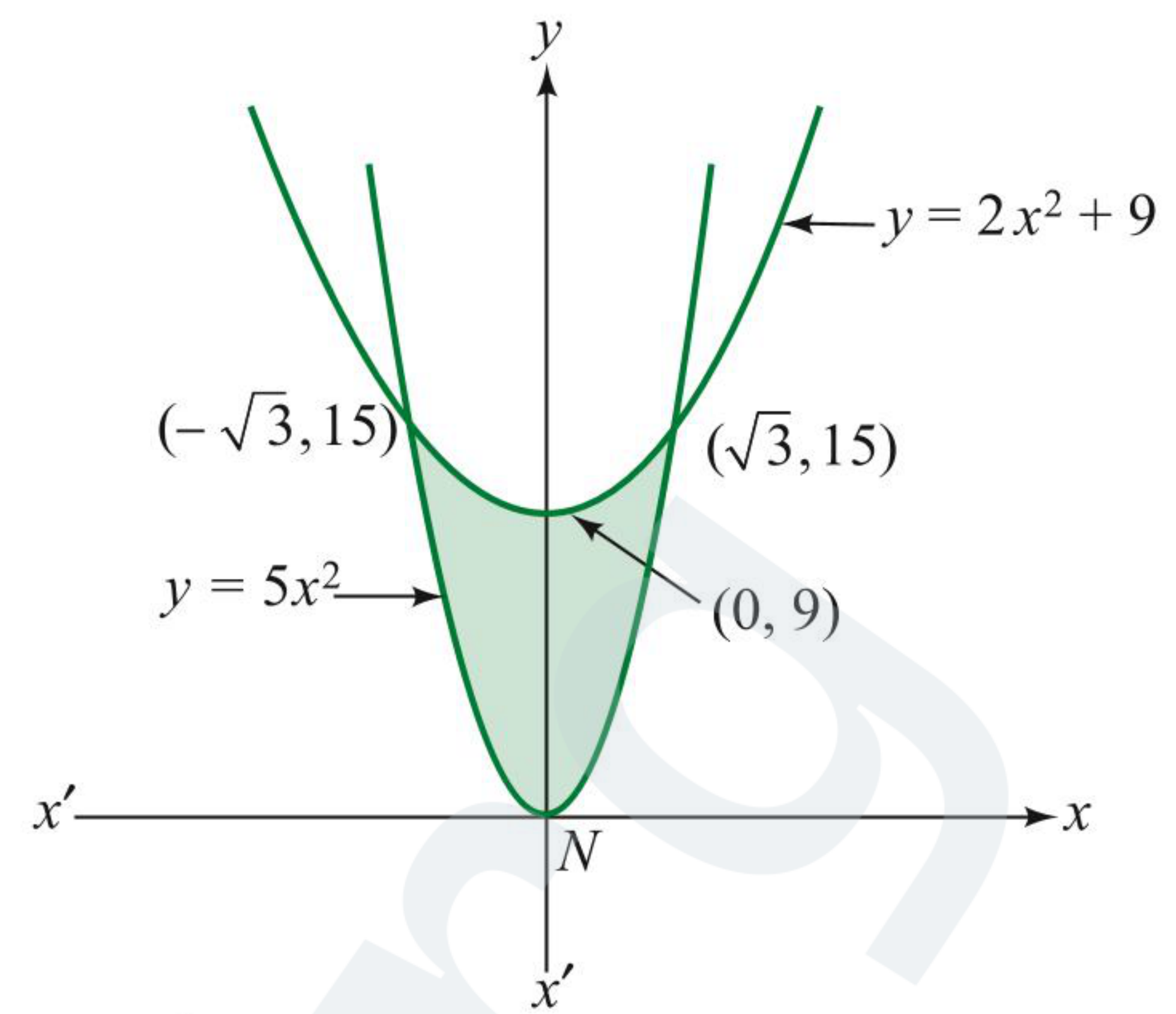
9. (1) $y = x(1 - \log_e x)$

It meets x -axis, if $x(1 - \log_e x) = 0$

$$\therefore x = 0 \text{ or } x = e$$

$$\begin{aligned}
 \therefore \text{ Required area} &= \int_0^e x(1 - \log_e x)dx \\
 &= \int_0^e xdx - \int_0^e x \log_e x dx \\
 &= \left[\frac{x^2}{2} \right]_0^e - \left[\frac{x^2}{2} \log_e x \right]_0^e + \int_0^e \frac{x}{2} dx \\
 &= \frac{e^2}{2} - \left[\frac{e^2}{2} - \lim_{x \rightarrow 0} \frac{x^2}{2} \log_e x \right] + \frac{e^2}{4} \\
 &= \frac{e^2}{4}
 \end{aligned}$$

10. (1)



$$\text{Given } 5x^2 - y = 0 \quad \dots(1)$$

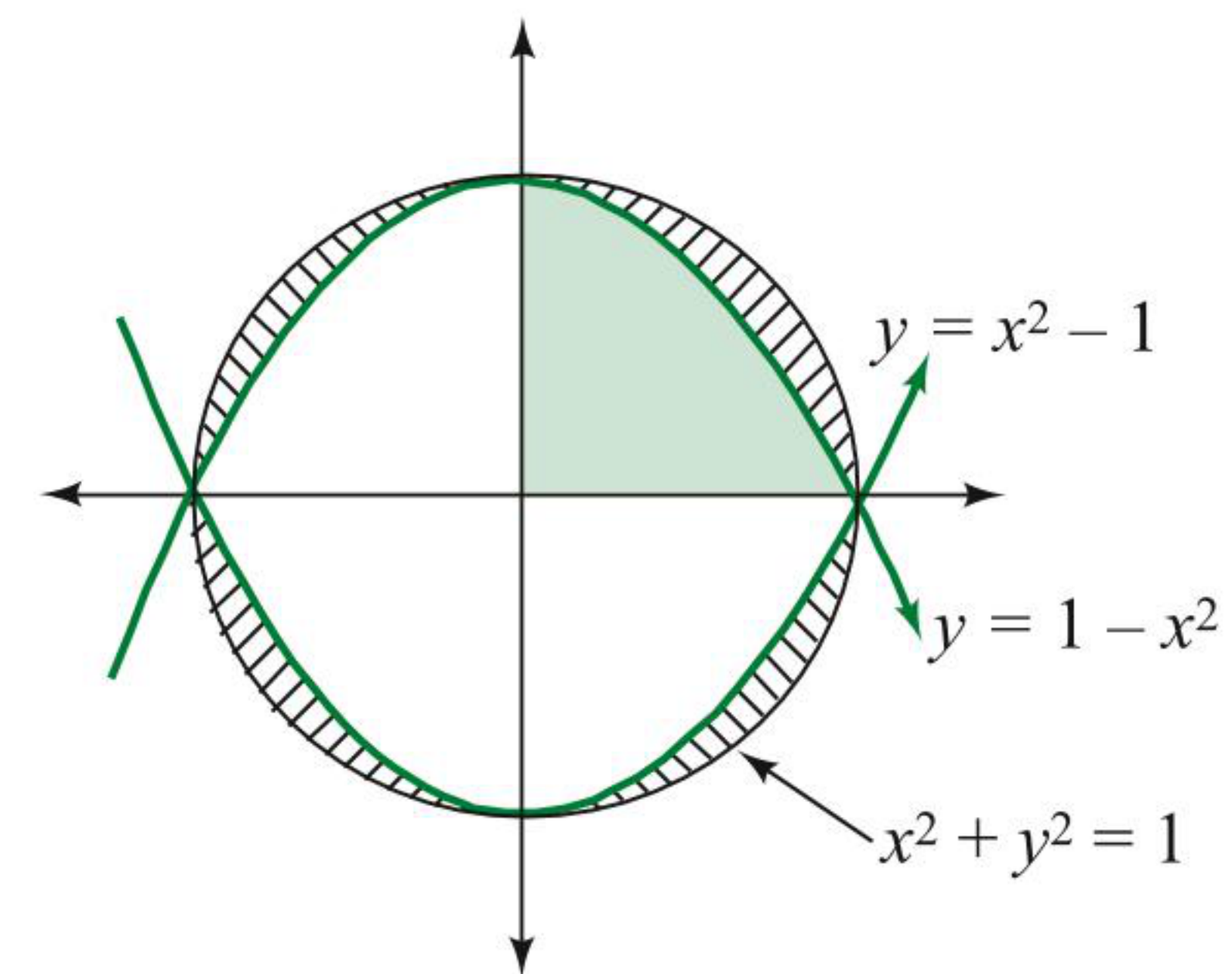
$$\text{and } 2x^2 - y + 9 = 0 \quad \dots(2)$$

Eliminating y , we get

$$\begin{aligned}
 5x^2 - (2x^2 + 9) &= 0 \\
 \text{or } 3x^2 &= 9 \text{ or } x = -\sqrt{3}, \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ Required area} &= 2 \int_0^{\sqrt{3}} ((2x^2 + 9) - 5x^2) dx \\
 &= 2 \int_0^{\sqrt{3}} (9 - 3x^2) dx \\
 &= 2 \left[9x - x^3 \right]_0^{\sqrt{3}} \\
 &= 2[9\sqrt{3} - 3\sqrt{3}] \\
 &= 12\sqrt{3} \text{ sq. units}
 \end{aligned}$$

11. (1)



The dotted area is

$$A = \int_0^1 (1 - x^2)dx = \left(x - \frac{x^3}{3} \right)_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence, area bounded by circle $x^2 + y^2 = 1$ and

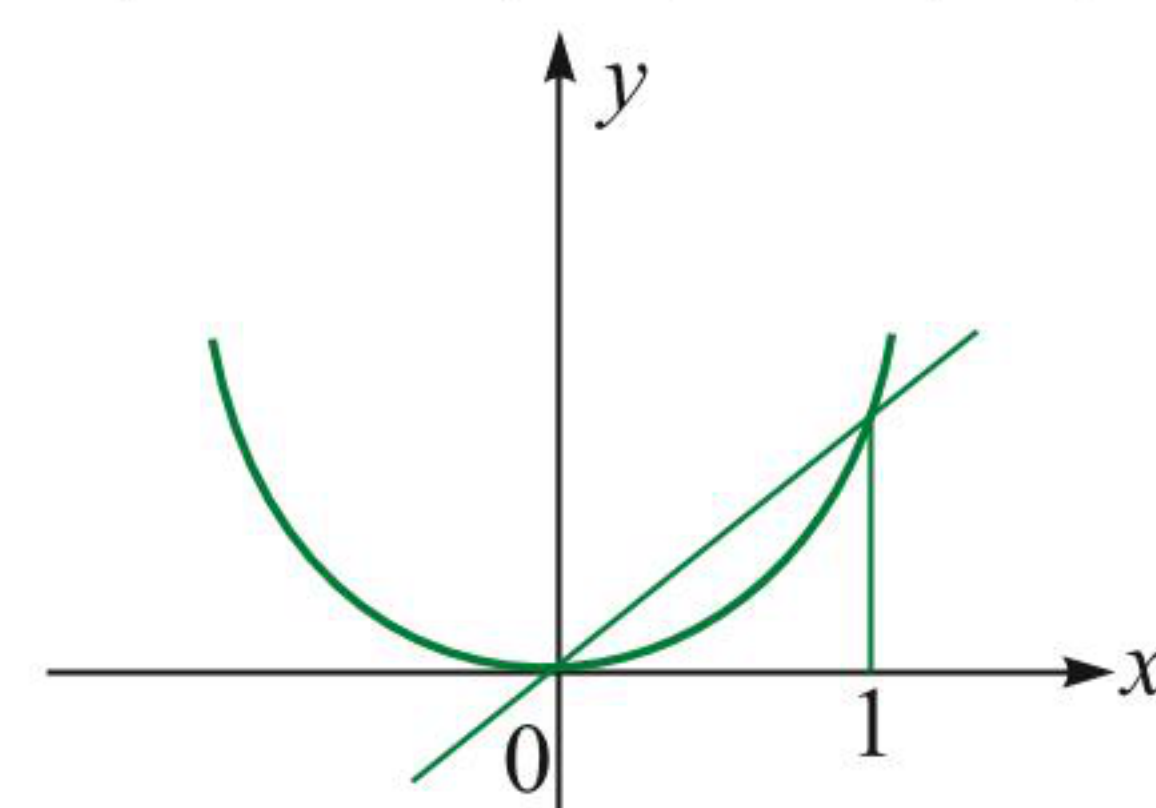
$$|y| = 1 - x^2$$

= Lined area

= Area of circle - Area bounded by $|y| = 1 - x^2$

$$= \pi - 4 \times \left(\frac{2}{3} \right) = \frac{3\pi - 8}{3} \text{ sq. units.}$$

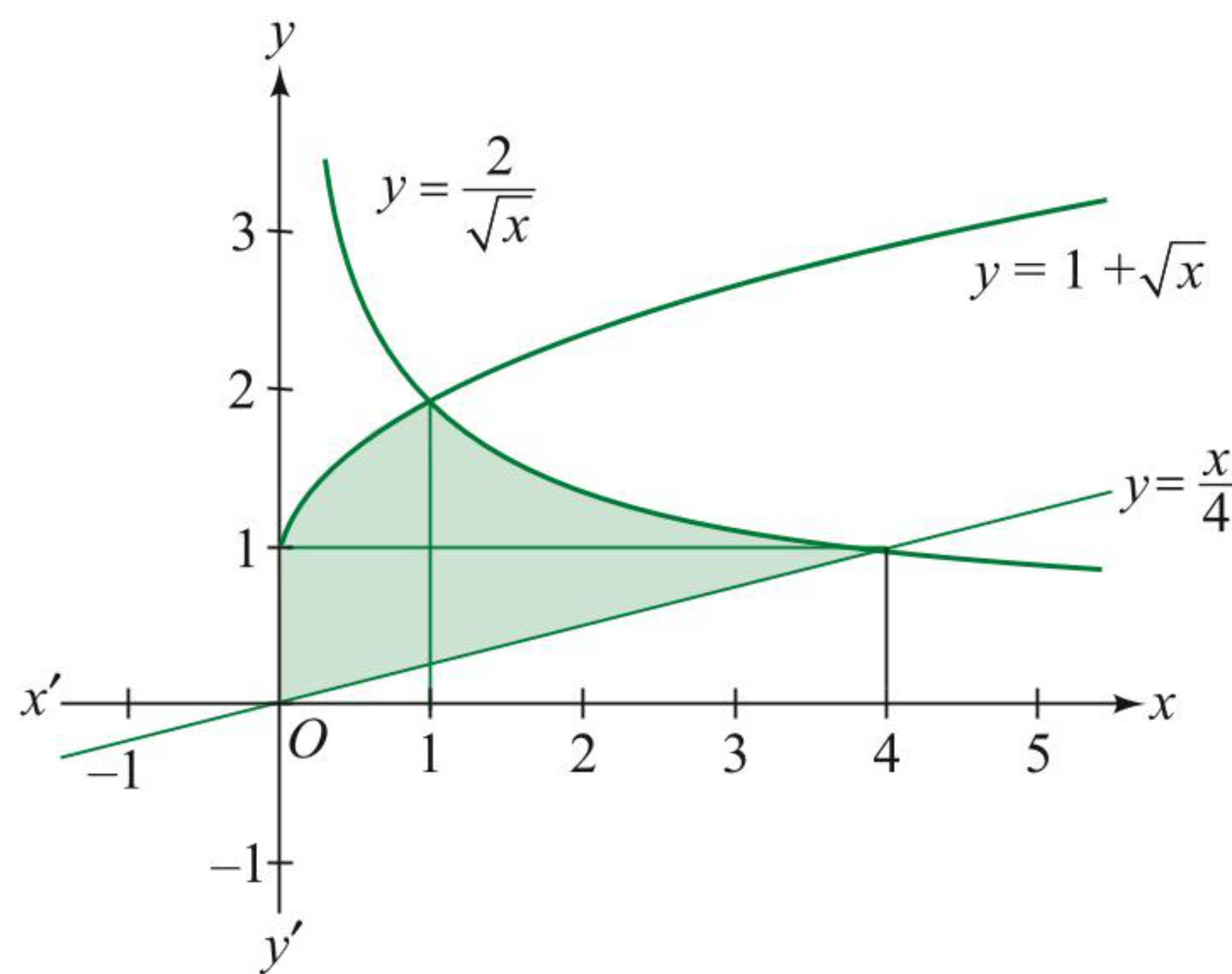
12. (4) $y = x$ intersect $y = x^n$ at $(0, 0)$ and $(1, 1)$ for all $n \in \mathbb{N}$



$$\text{Area } A_n = \int_0^1 (x - x^n) dx$$

$$\begin{aligned}
 &= \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1} \right]_0^1 \\
 &= \frac{1}{2} - \frac{1}{n+1} \\
 &= \frac{n-1}{2(n+1)} \\
 \text{Thus, } A_2 \cdot A_3 \cdot A_4 \cdots A_n &= \frac{1}{2^{n-1}} \left(\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdots \frac{n-1}{n+1} \right) \\
 &= \frac{1}{2^{n-2} \cdot n(n+1)}
 \end{aligned}$$

13. (3)

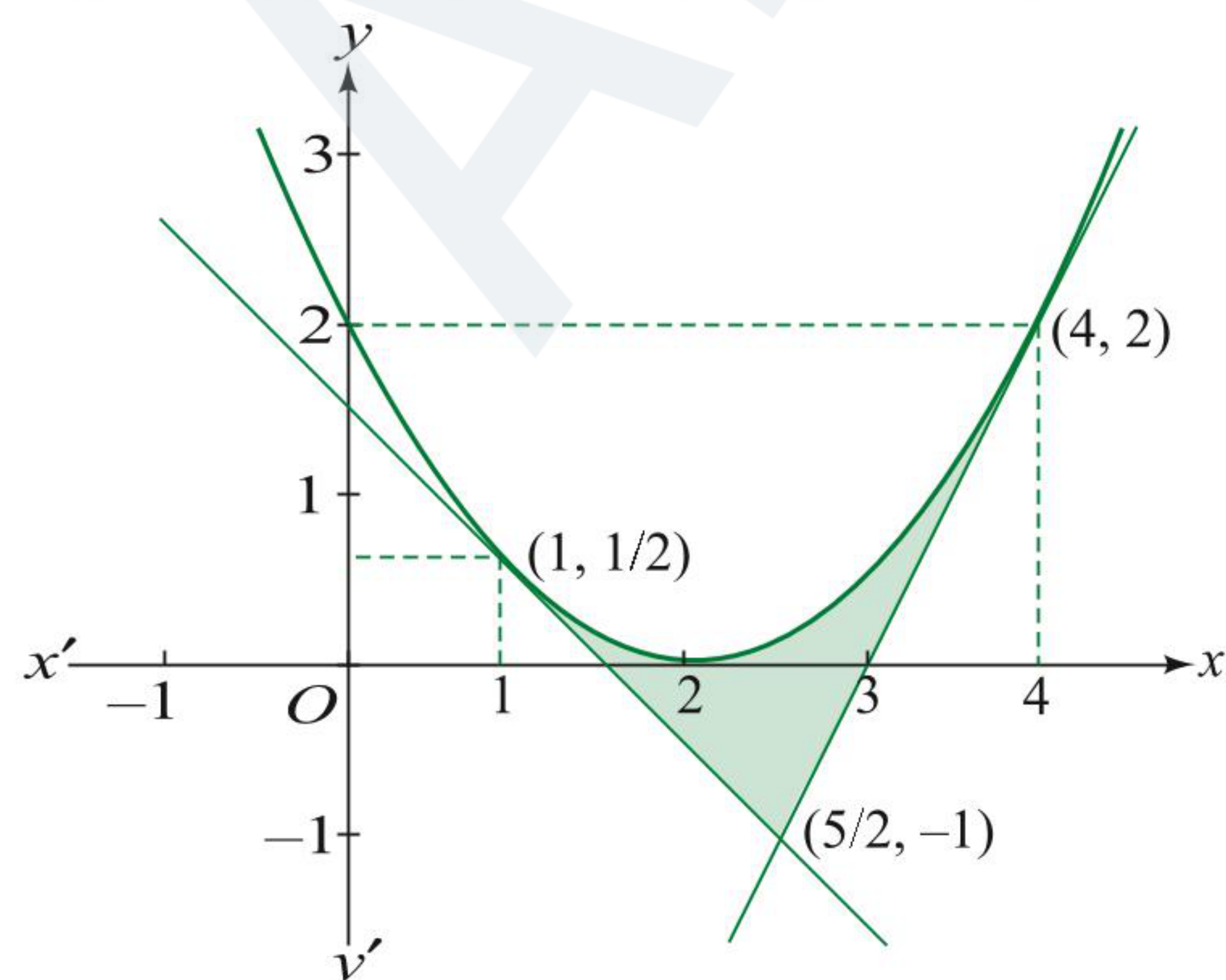


$$\begin{aligned}
 A_1 &= \int_0^1 \left(1 + \sqrt{x} - \frac{x}{4} \right) dx \\
 &= \left[x + \frac{2x^{3/2}}{3} - \frac{x^2}{8} \right]_0^1 = 1 + \frac{2}{3} - \frac{1}{8} = \frac{37}{24} \\
 A_2 &= \int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{x}{4} \right) dx \\
 &= \left[4\sqrt{x} - \frac{x^2}{8} \right]_1^4 \\
 &= \left[8 - 2 - 4 + \frac{1}{8} \right] = \frac{17}{8}
 \end{aligned}$$

$$\text{or } A = A_1 + A_2 = \frac{88}{24} = \frac{11}{3} \text{ sq. units}$$

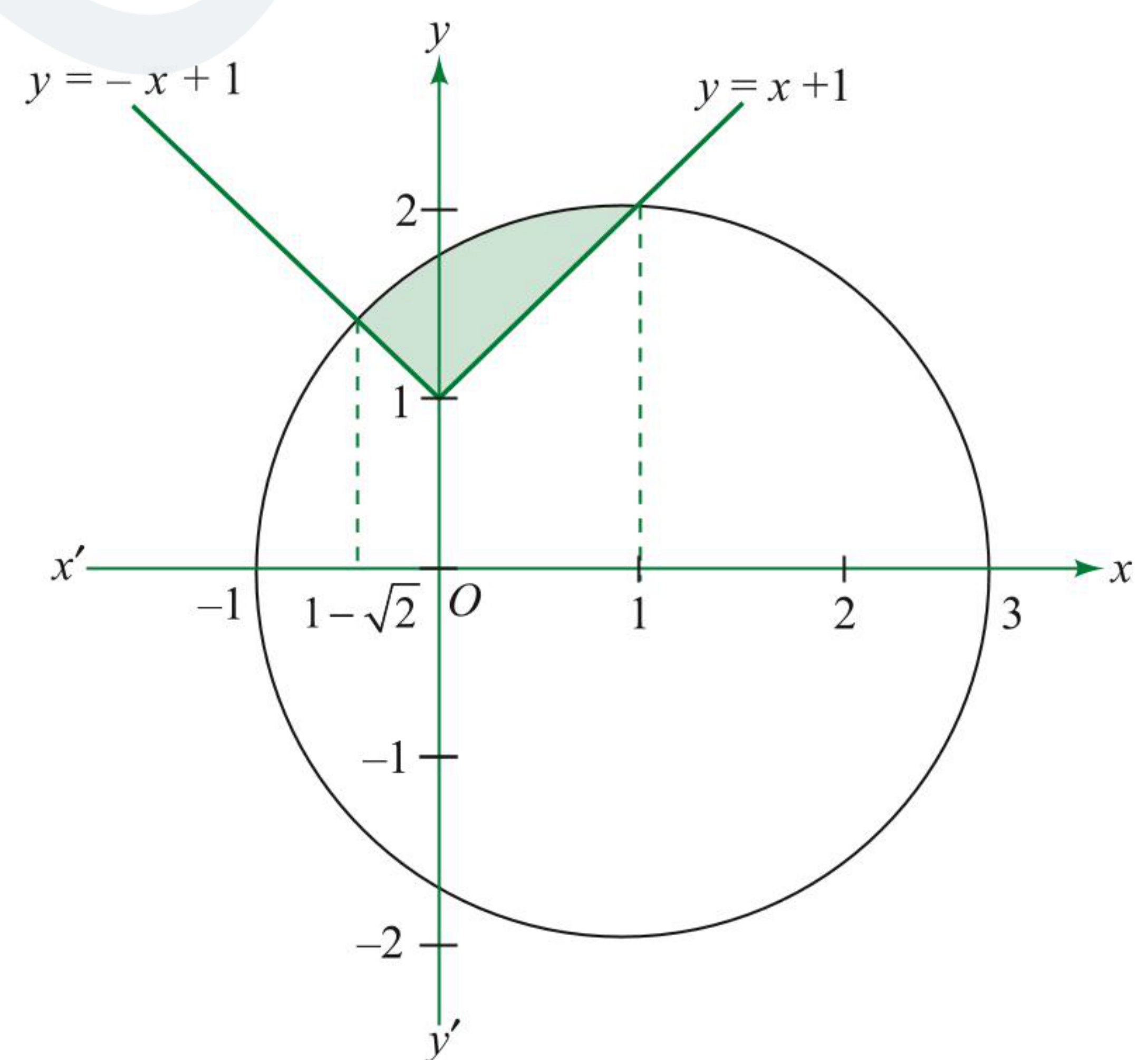
$$\begin{aligned}
 \text{14. (1) } y &= \frac{x^2}{2} - 2x + 2 = \frac{(x-2)^2}{2}, \\
 \frac{dy}{dx} &= x-2, \quad \left(\frac{dy}{dx} \right)_{x=1} = -1, \quad \left(\frac{dy}{dx} \right)_{x=4} = 2
 \end{aligned}$$

Thus, tangent at $(1, 1/2)$ is $y - 1/2 = -1(x - 1)$ or $2x + 2y - 3 = 0$
 Tangent at $(4, 2)$ is $y - 2 = 2(x - 4)$ or $2x - y - 6 = 0$



$$\begin{aligned}
 \text{Hence, } A &= \int_1^{5/2} \left(\frac{x^2}{2} - 2x + 2 - \frac{3-2x}{2} \right) dx \\
 &\quad + \int_{5/2}^4 \left(\frac{x^2}{2} - 2x + 2 - (2x-6) \right) dx \\
 &= \int_1^{5/2} \left(\frac{x^2}{2} - 2x + 2 \right) dx - \int_1^{5/2} \left(\frac{3-2x}{2} \right) dx - \int_{5/2}^4 (2x-6) dx \\
 &= \left(\frac{x^3}{6} - x^2 + 2x \right)_1^{5/2} - \frac{1}{2} (3x - x^2)_1^{5/2} - (x^2 - 6x)_{5/2}^4 \\
 &= \left(\frac{63}{6} - 15 + 6 \right) - \frac{1}{2} \left(3 \times \frac{5}{2} - \left(\frac{25}{4} - 1 \right) \right) \\
 &\quad - \left(\left(16 - \frac{25}{4} \right) - 6 \left(4 - \frac{5}{2} \right) \right) \\
 &= \frac{3}{2} - \frac{1}{2} \left(\frac{9}{2} - \frac{21}{4} \right) - \left(\frac{39}{4} - 6 \left(\frac{3}{2} \right) \right) \\
 &= \frac{9}{8} \text{ sq. units}
 \end{aligned}$$

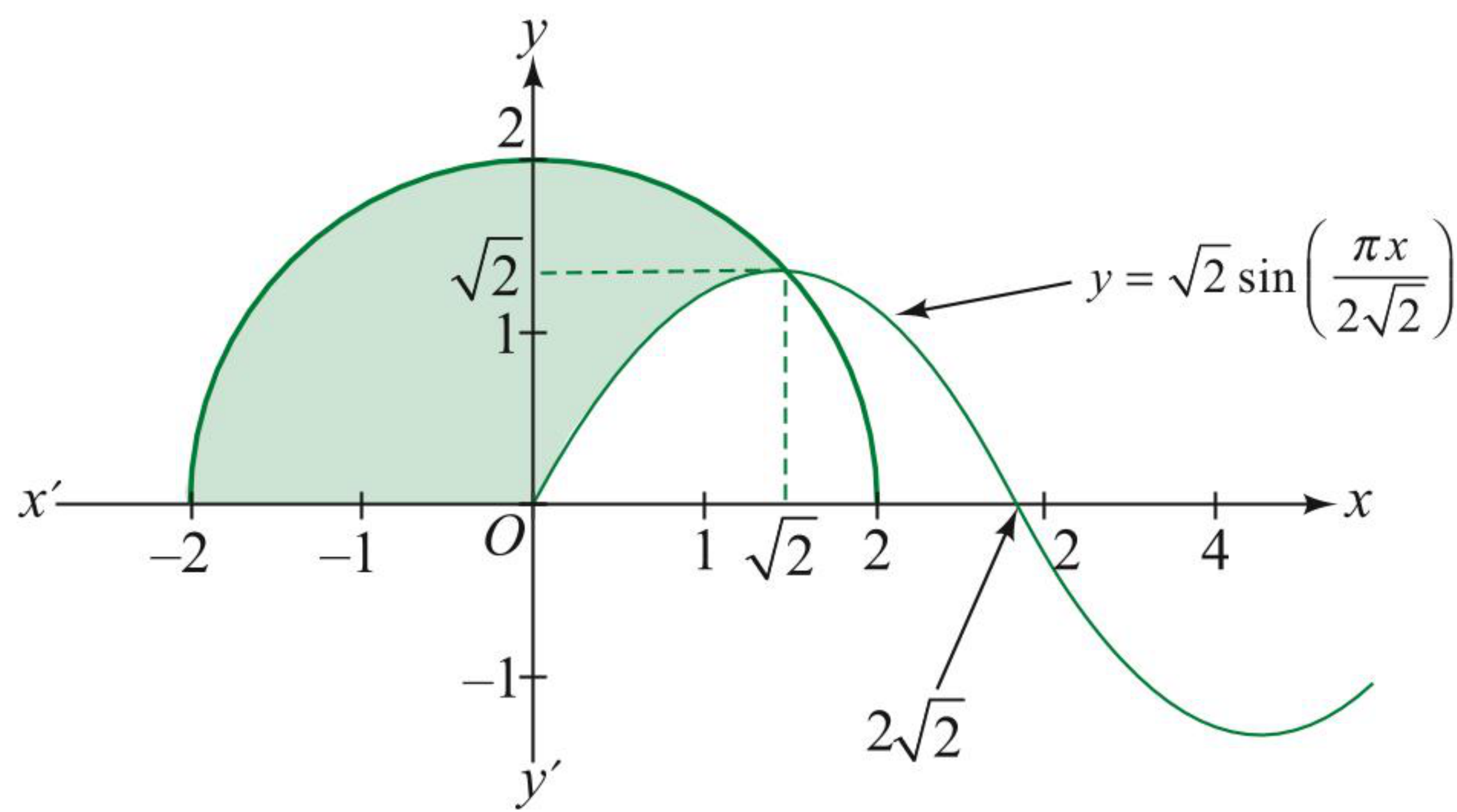
15. (1)



$$\begin{aligned}
 &x^2 + y^2 - 2x - 3 = 0 \\
 \text{or } &(x-1)^2 + y^2 = 4
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_{1-\sqrt{2}}^0 (\sqrt{4-(x-1)^2} - (-x+1)) dx \\
 &\quad + \int_0^1 (\sqrt{4-(x-1)^2} - (x+1)) dx \\
 &= \frac{x-1}{2} \sqrt{4-(x-1)^2} + \frac{4}{2} \sin^{-1} \frac{x-1}{2} + \frac{x^2}{2} - x \Big|_{1-\sqrt{2}}^0 \\
 &\quad + \frac{x-1}{2} \sqrt{4-(x-1)^2} + \frac{4}{2} \sin^{-1} \frac{x-1}{2} - \frac{x^2}{2} - x \Big|_0^1 \\
 &= \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3} \right) - \left(\frac{-\sqrt{2}}{2} \sqrt{2} - \frac{\pi}{2} + \frac{3-2\sqrt{2}}{2} - 1 + \sqrt{2} \right) \\
 &\quad + \left(-\frac{1}{2} - 1 \right) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3} \right) \\
 &= -\left(-1 - \frac{\pi}{2} + \frac{3}{2} - \sqrt{2} - 1 + \sqrt{2} \right) - \frac{3}{2} \\
 &= \frac{\pi}{2} - 1 \text{ sq. units.}
 \end{aligned}$$

16. (4) $y = \sqrt{4-x^2}$, $y = \sqrt{2} \sin\left(\frac{x\pi}{2\sqrt{2}}\right)$ intersect at $x = \sqrt{2}$



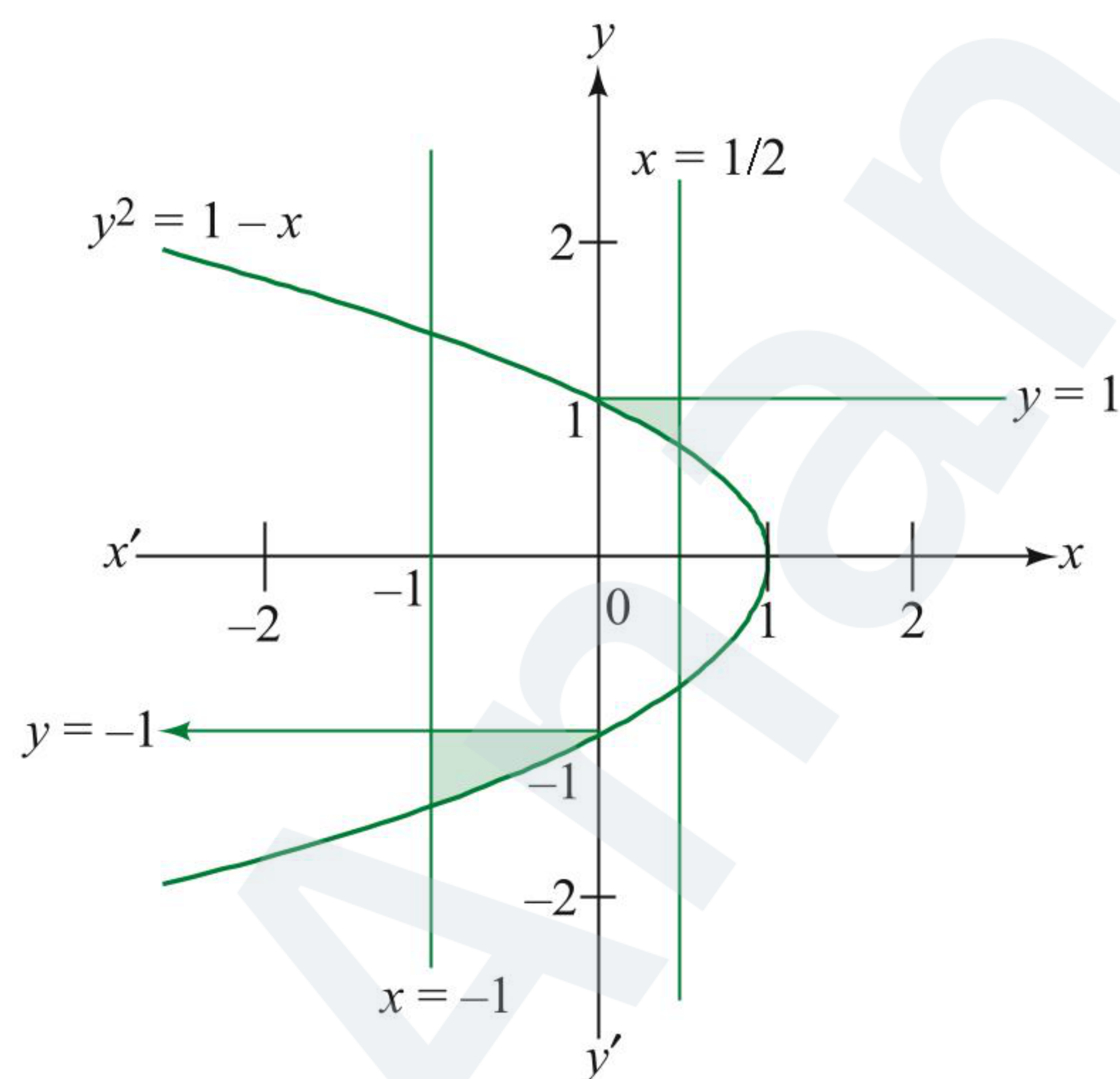
Area to the left of y -axis is π

Area to the right of y -axis

$$\begin{aligned} &= \int_0^{\sqrt{2}} \left(\sqrt{4-x^2} - \sqrt{2} \sin \frac{x\pi}{2\sqrt{2}} \right) dx \\ &= \left(\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right)_0^{\sqrt{2}} + \left(\frac{4}{\pi} \cos \frac{x\pi}{2\sqrt{2}} \right)_0^{\sqrt{2}} \\ &= \left(1 + 2 \times \frac{\pi}{4} \right) + \frac{4}{\pi} (0-1) \\ &= 1 + \frac{\pi}{2} - \frac{4}{\pi} \\ &= \frac{2\pi + \pi^2 - 8}{2\pi} \text{ sq. units.} \end{aligned}$$

$$\therefore \text{Ratio} = \frac{2\pi^2}{2\pi + \pi^2 - 8}.$$

17. (1)

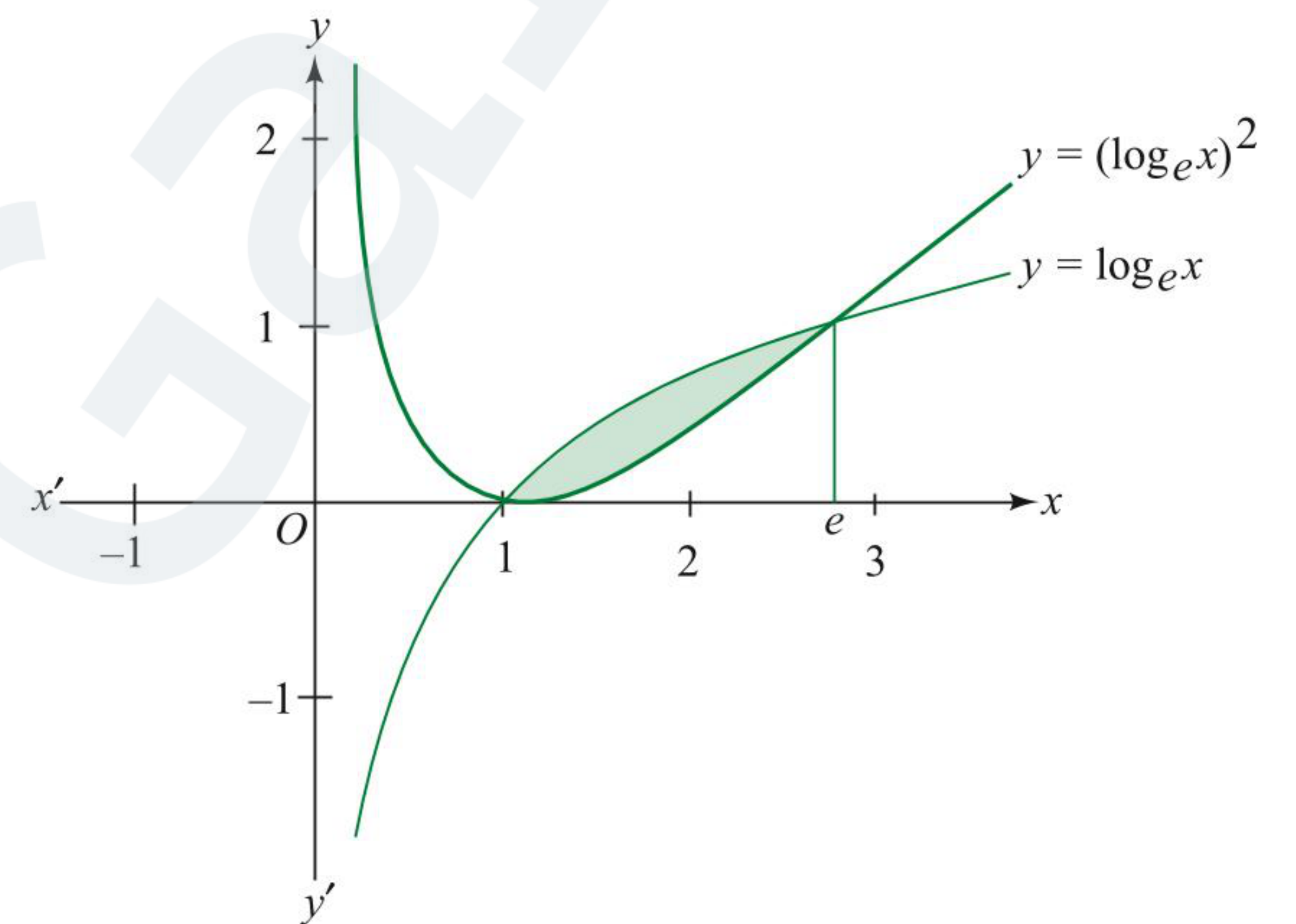


From the figure,

$$\begin{aligned} A &= \int_{-1}^0 \left(-1 - (-\sqrt{1-x}) \right) dx + \int_0^{1/2} (1 - \sqrt{1-x}) dx \\ &= \left[-x - \frac{(1-x)^{3/2}}{3/2} \right]_{-1}^0 + \left[x + \frac{(1-x)^{3/2}}{3/2} \right]_0^{1/2} \\ &= \left[-\frac{2}{3} - \left(1 - \frac{2 \times 2^{3/2}}{3} \right) \right] + \left[\frac{1}{2} + \frac{2}{3 \times 2^{3/2}} - \frac{2}{3} \right] \end{aligned}$$

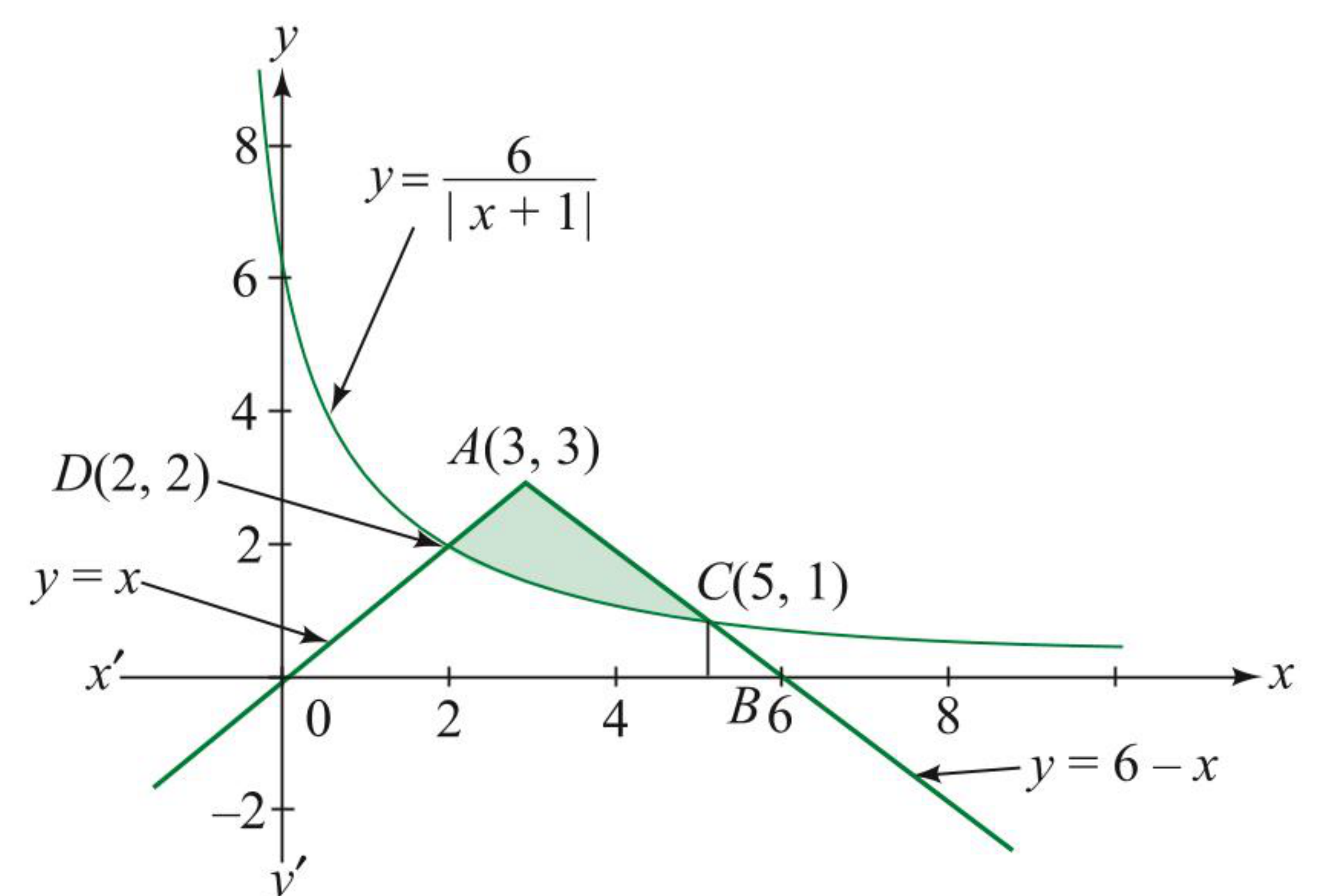
$$\begin{aligned} &= \frac{2}{3 \times 2^{3/2}} + \frac{2 \times 2^{3/2}}{3} - \frac{4}{3} - \frac{1}{2} \\ &= \frac{3}{\sqrt{2}} - \frac{4}{3} - \frac{1}{2} \\ &= \frac{3}{\sqrt{2}} - \frac{11}{6} \text{ sq. units} \end{aligned}$$

18. (2) Given curves are $y = \log_e x$ and $y = (\log_e x)^2$
Solving $\log_e x = (\log_e x)^2 \Rightarrow \log_e x = 0, 1 \Rightarrow x = 1$ and $x = e$
Also, for $1 < x < e$, $0 < \log_e x < 1 \Rightarrow \log_e x > (\log_e x)^2$
For $x > e$, $\log_e x < (\log_e x)^2$
 $y = (\log_e x)^2 > 0$ for all $x > 0$
and when $x \rightarrow 0$, $(\log_e x)^2 \rightarrow \infty$.
From these information, we can plot the graph of the functions.



$$\begin{aligned} \therefore \text{Required area} &= \int_1^e (\log x - (\log_e x)^2) dx \\ &= \int_1^e \log x dx - \int_1^e (\log_e x)^2 dx \\ &= [x \log_e x - x]_1^e - [x(\log_e x)^2]_1^e + \int_1^e \frac{2 \log_e x}{x} x dx \\ &= 1 - e + 2[x \log_e x - x]_1^e \\ &= 3 - e \text{ sq. units.} \end{aligned}$$

19. (3)



First consider $y = 3 - |3 - x|$
For $x < 3$; $y = 3 - (3 - x) = x$
For $x \geq 3$; $y = 3 - (x - 3) = 6 - x$
Consider $y = \frac{6}{|x+1|}$
For $x < -1$; $y = \frac{6}{-1-x}$

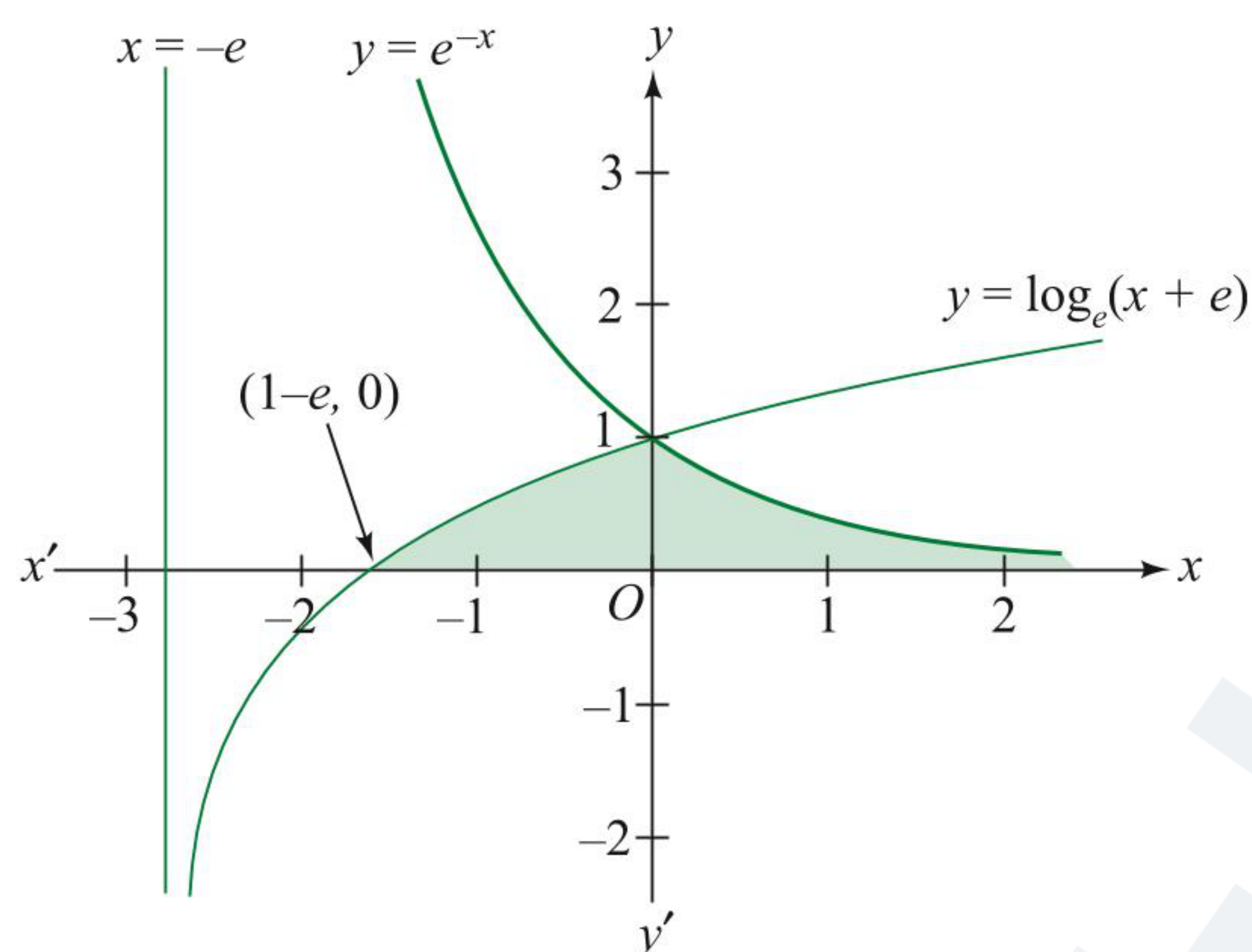
$$\Rightarrow (1+x)y = -6$$

$$\text{For } x > -1; \quad y = \frac{6}{x+1}$$

$$\begin{aligned} \text{Required area} &= \left[\int_2^3 \left(x - \frac{6}{x+1} \right) dx + \int_3^5 \left((6-x) - \frac{6}{x+1} \right) dx \right] \\ &= \left[\left(\frac{x^2}{2} \right)_2^3 + \left(6x - \frac{x^2}{2} \right)_3^5 - (6 \log(x+1))_2^5 \right] \\ &= \left[\frac{5}{2} + 4 - 6 \log 2 \right] \\ &= \frac{13}{2} - 6 \ln 2 \text{ sq. units.} \end{aligned}$$

20. (1) $y = \log_e(x+e)$, $x = \log_e\left(\frac{1}{y}\right)$ or $y = e^{-x}$.

For $y = \log_e(x+e)$, shift the graph of $y = \log_e x$, e units to the left hand side.



$$\begin{aligned} \text{Required area} &= \int_{1-e}^0 \log_e(x+e) dx + \int_0^\infty e^{-x} dx \\ &= |x \log_e(x+e)|_{1-e}^0 - \int_{1-e}^0 \frac{x}{x+e} dx - |e^{-x}|_0^\infty \\ &= \int_0^{1-e} \left(1 - \frac{e}{x+e} \right) dx - e^{-\infty} + e^0 \\ &= |x - e \log(x+e)|_0^{1-e} - 0 + 1 \\ &= 1 - e + e \log e + 1 = 2 \text{ sq. units.} \end{aligned}$$

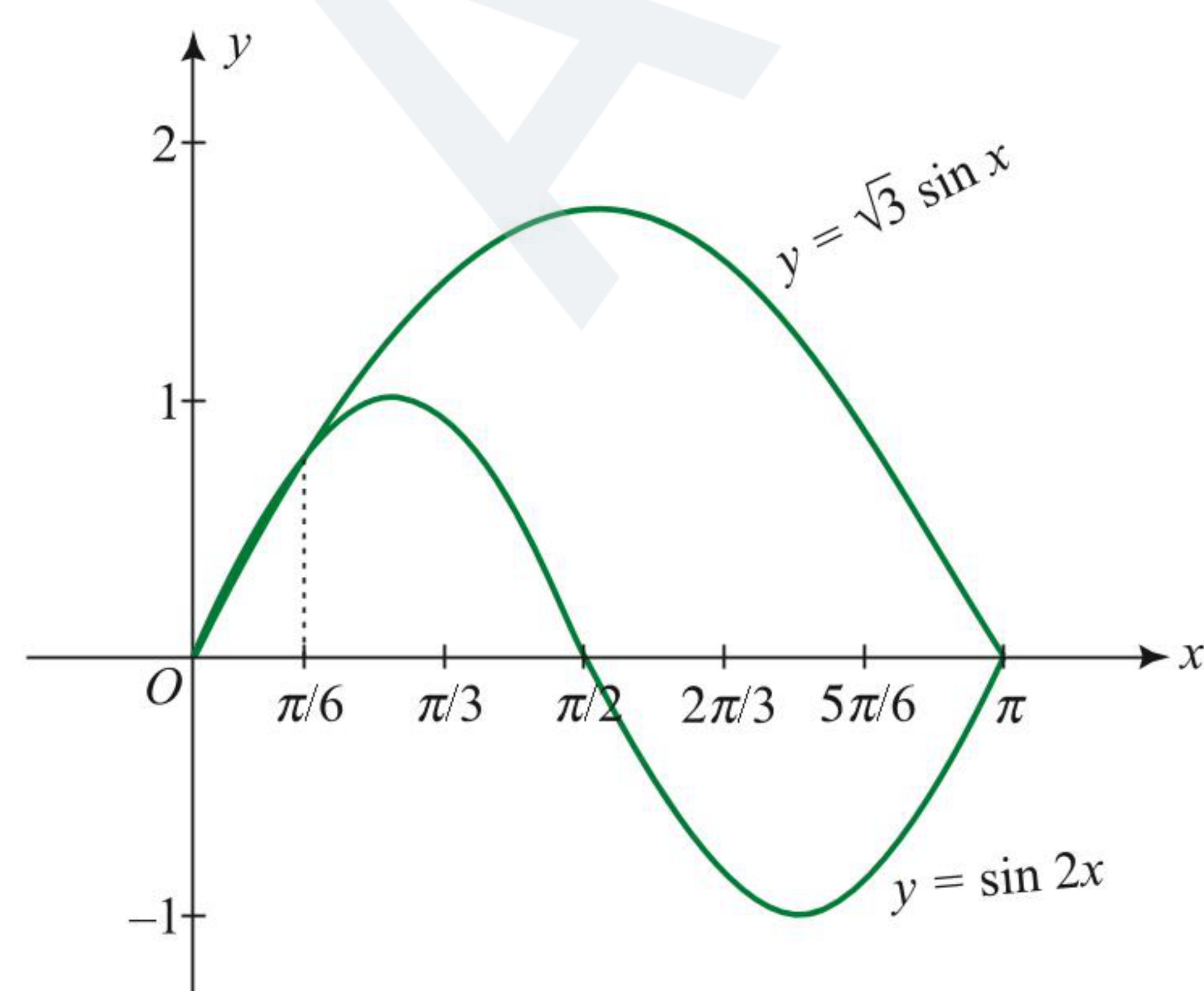
21. (1) We have $y = \sin 2x$, $y = \sqrt{3} \sin x$

Solving, we get $\sin 2x = \sqrt{3} \sin x$

$$\therefore 2 \sin x \cos x = \sqrt{3} \sin x$$

$$\therefore \sin x = 0 \text{ or } \cos x = \sqrt{3}/2$$

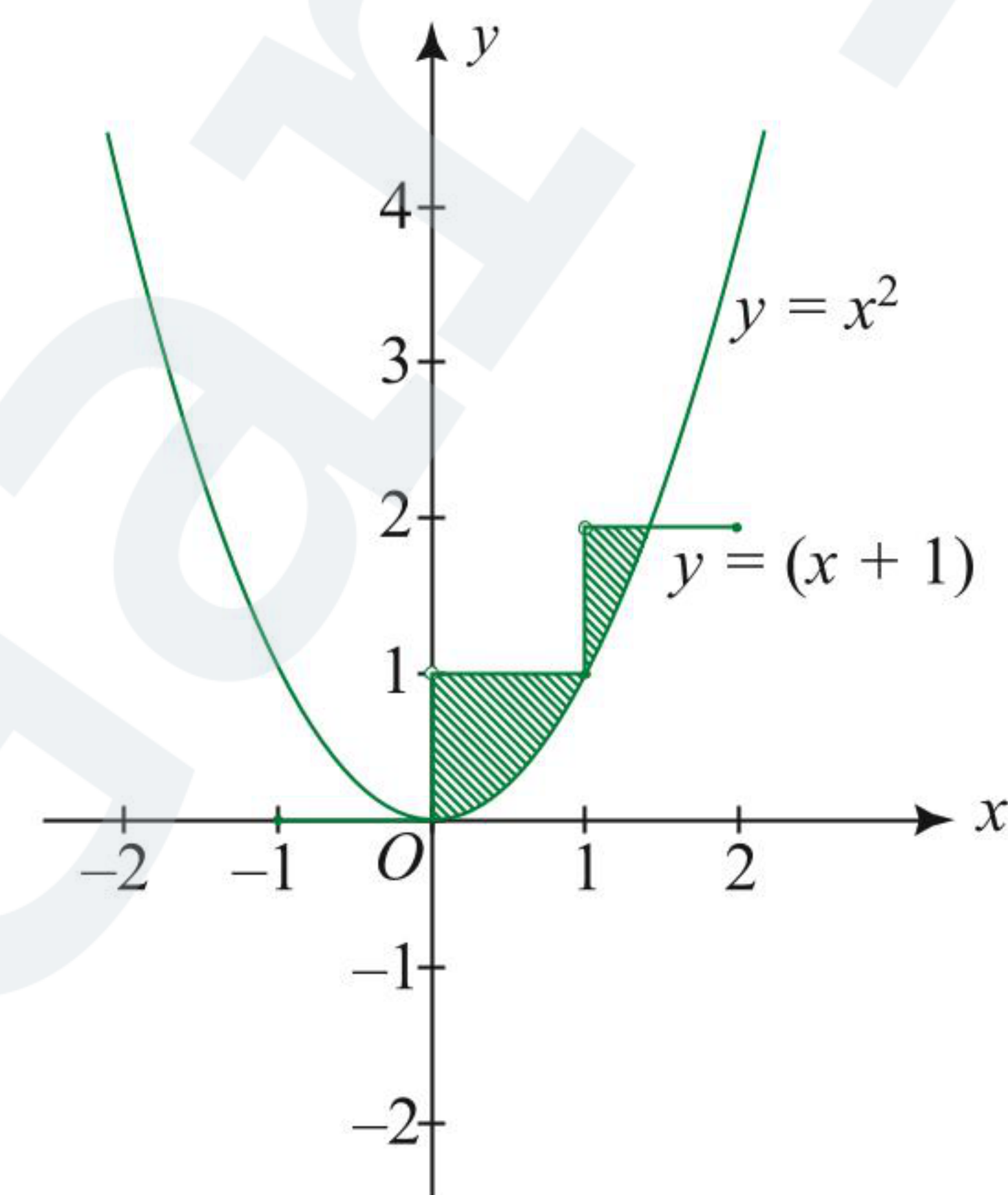
$$\therefore x = 0, \pi \text{ or } x = \pi/6$$



$\therefore$ Required area

$$\begin{aligned} &= \int_0^{\pi/6} (\sin 2x - \sqrt{3} \sin x) dx + \int_{\pi/6}^\pi (\sqrt{3} \sin x - \sin 2x) dx \\ &= \left[-\frac{\cos 2x}{2} + \sqrt{3} \cos x \right]_0^{\pi/6} + \left[-\sqrt{3} \cos x + \frac{\cos 2x}{2} \right]_{\pi/6}^\pi \\ &= \left[\frac{7}{4} - \sqrt{3} \right] + \left[\sqrt{3} + \frac{7}{4} \right] \\ &= \frac{7}{2} \end{aligned}$$

22. (2)



$$\begin{aligned} \text{Area} \int_0^1 (1-x^2) dx + \int_1^{\sqrt{2}} (1-x^2) dx &= \left[x - \frac{x^3}{3} \right]_0^1 + \left[x - \frac{x^3}{3} \right]_1^{\sqrt{2}} \\ &= \frac{\sqrt{2}}{3} \end{aligned}$$

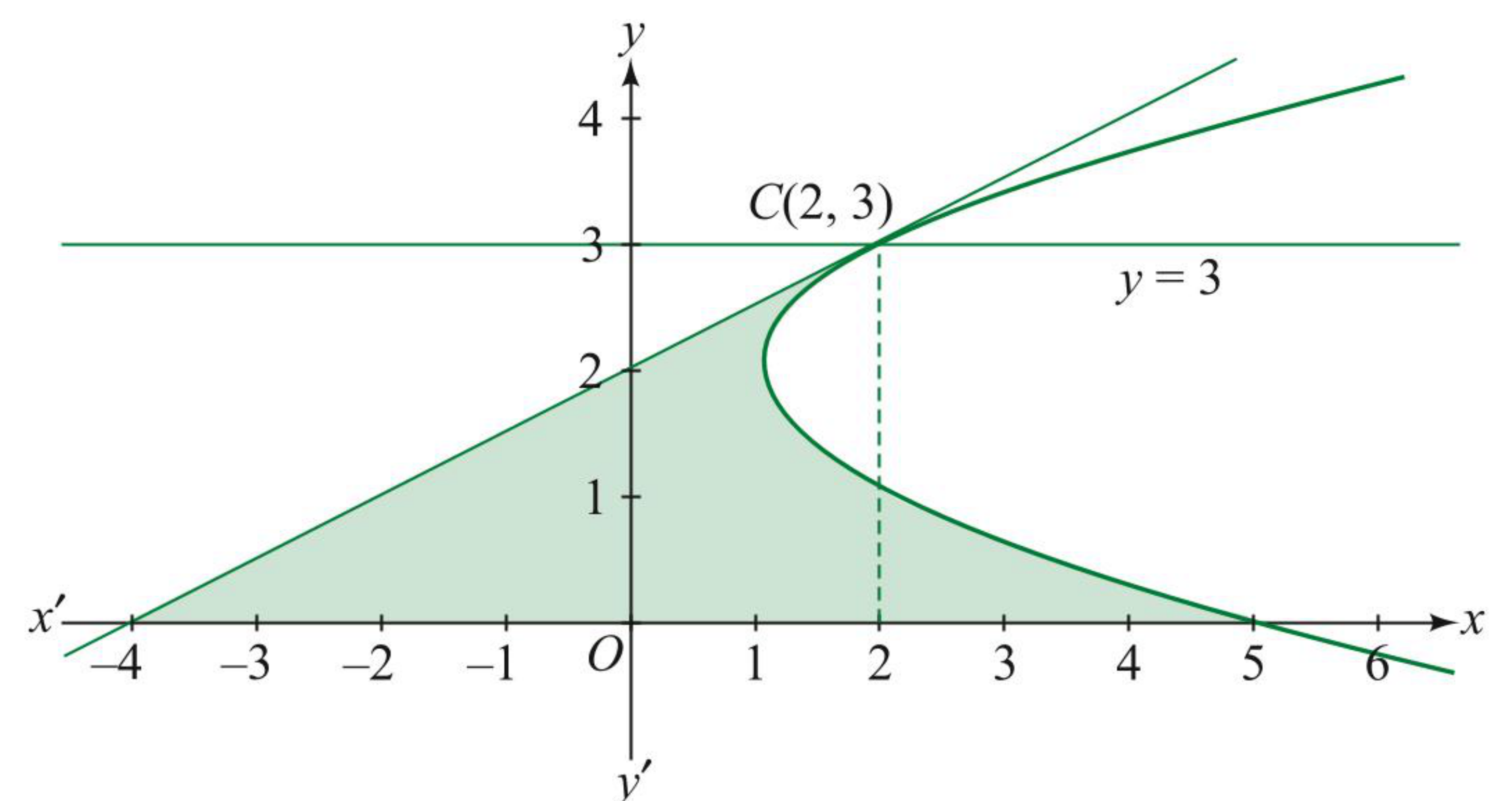
23. (3) Given parabola is $(y-2)^2 = x-1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$$

When $y = 3$, $x = 2$

$$\therefore \frac{dy}{dx} = \frac{1}{2(3-2)} = \frac{1}{2}$$

Tangent at $(2, 3)$ is $y-3 = \frac{1}{2}(x-2)$ or $x-2y+4=0$



$$\begin{aligned} \therefore \text{Required area} &= \int_0^3 ((y-2)^2 + 1) dy - \int_0^3 (2y-4) dy \\ &= \left| \frac{(y-2)^3}{3} + y \right|_0^3 - |y^2 - 4y|_0^3 \\ &= \frac{1}{3} + 3 + \frac{8}{3} - (9 - 12) = 9 \text{ sq. units.} \end{aligned}$$

24. (1) Curve tracing : $y = x e^x$

Let $\frac{dy}{dx} = 0 \Rightarrow e^x + x e^x = 0$ or $x = -1$.

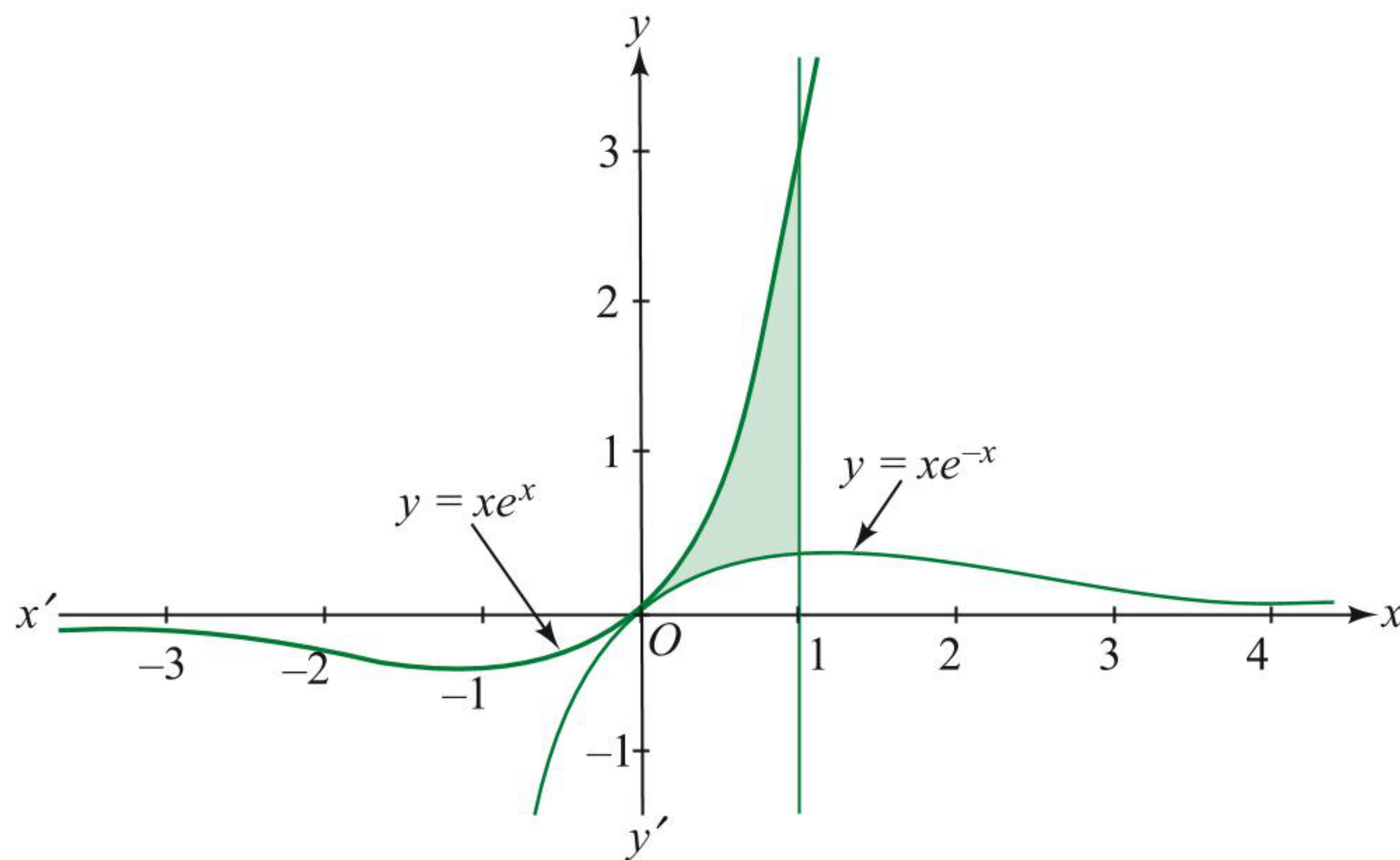
Also, at $x = -1$, $\frac{dy}{dx}$ changes sign from -ve to +ve,

Hence, $x = -1$ is a point of minima.

When $x \rightarrow \infty$, $y \rightarrow \infty$

$$\text{Also } \lim_{x \rightarrow -\infty} x e^x = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = 0$$

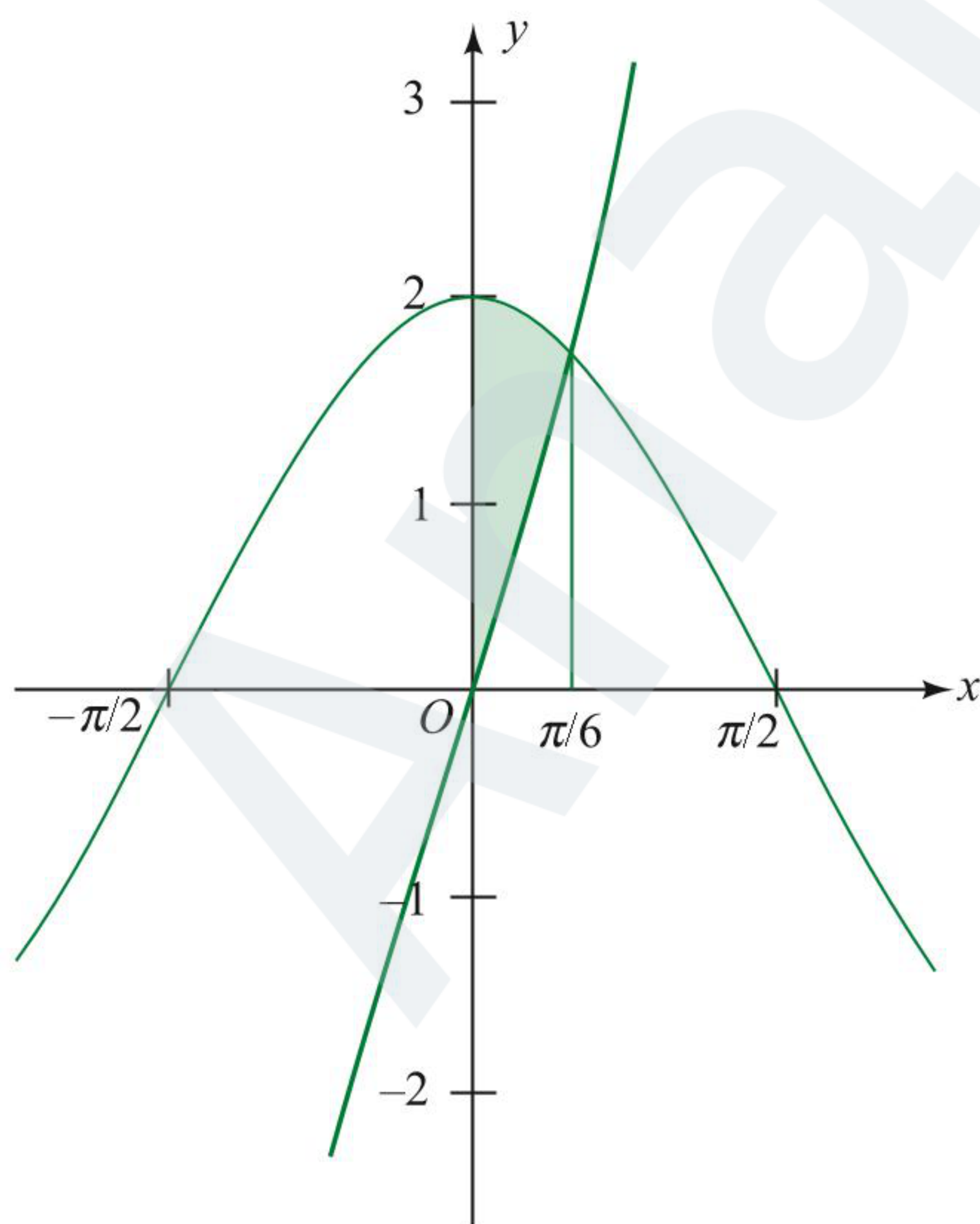
With similar types of arguments, we can draw the graph of $y = x e^{-x}$.



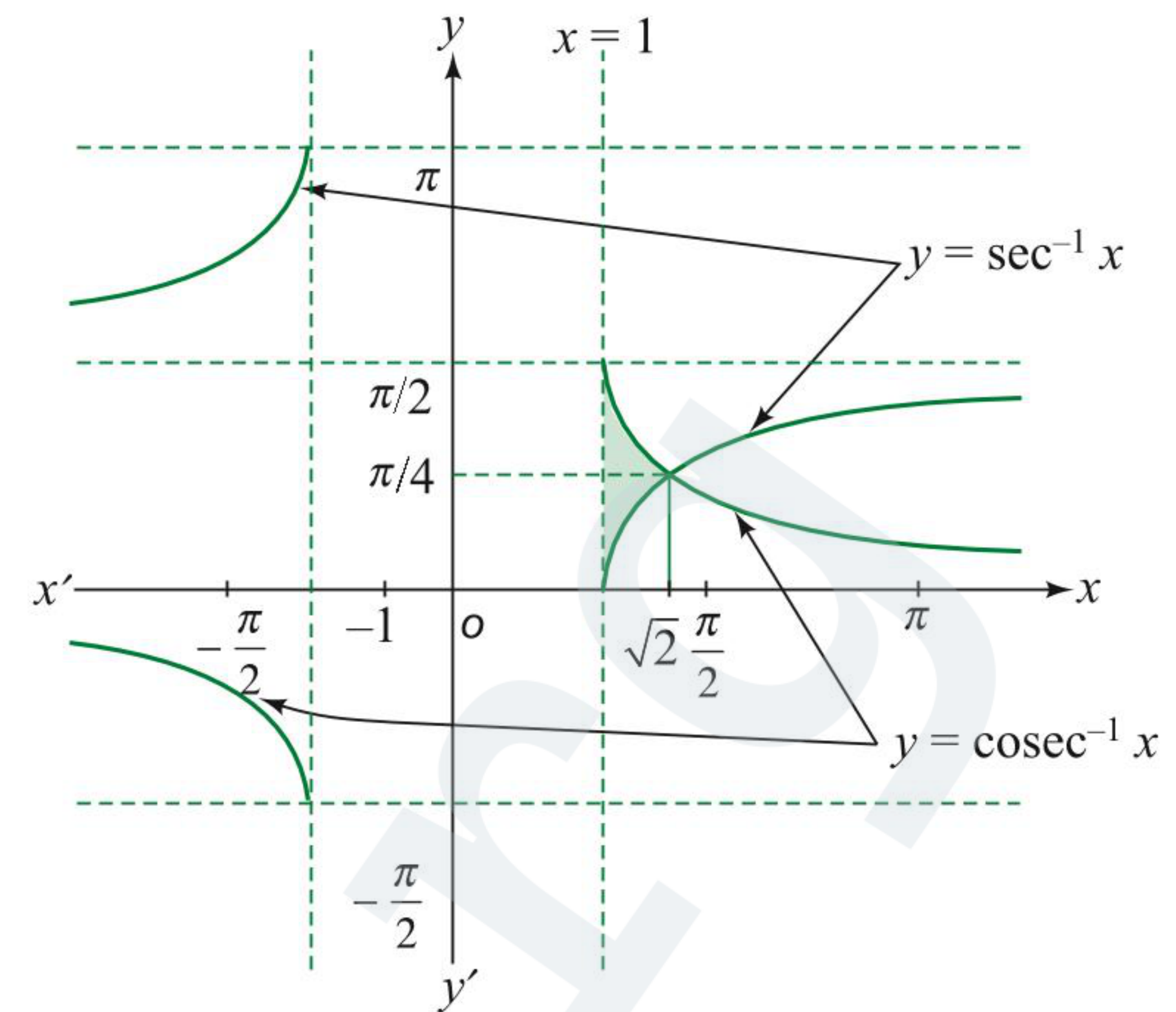
$$\begin{aligned} \text{Required area} &= \int_0^1 x e^x dx - \int_0^1 x e^{-x} dx \\ &= [x e^x]_0^1 - \int_0^1 e^x dx - \left([-x e^{-x}]_0^1 + \int_0^1 e^{-x} dx \right) \\ &= e - (e-1) - (-e^{-1} - (e^{-1} - 1)) = \frac{2}{e} \text{ sq. units.} \end{aligned}$$

25. (2) Solving $2 \cos x = 3 \tan x$, we get

$$2 - 2 \sin^2 x = 3 \sin x \text{ or } \sin x = \frac{1}{2} \text{ or } x = \frac{\pi}{6}.$$



$$\begin{aligned} \text{Required area} &= \int_0^{\pi/6} (2 \cos x - 3 \tan x) dx \\ &= 2 \sin x - 3 \log \sec x \Big|_0^{\pi/6} = 1 - 3 \ln 2 + \frac{3}{2} \ln 3 \text{ sq. units.} \end{aligned}$$

26. (1)

Integrating along x-axis, we get

$$A = \int_1^{\sqrt{2}} (\csc^{-1} x - \sec^{-1} x) dx$$

Integrating along y-axis, we get

$$\begin{aligned} A &= 2 \int_0^{\pi/4} (\sec y - 1) dy \\ &= 2 \left[\log |\sec y + \tan y| - y \right]_0^{\pi/4} \\ &= 2 \left[\log |\sqrt{2} + 1| - \frac{\pi}{4} \right] \\ &= \log (3 + 2\sqrt{2}) - \frac{\pi}{2} \text{ sq. units.} \end{aligned}$$

27. (4) $y^2 (2-x) = x^3$

$$\therefore y^2 = \frac{x^3}{2-x}$$

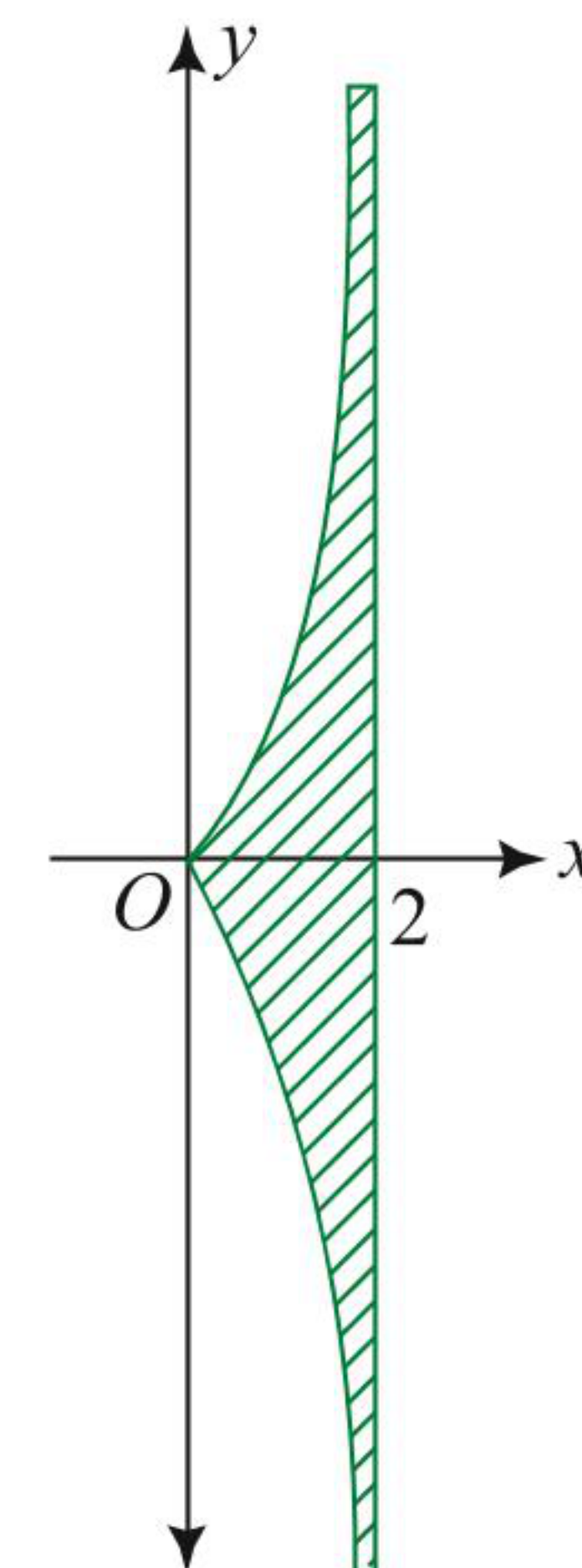
$$\therefore y = \pm \frac{x^{3/2}}{\sqrt{2-x}}, \text{ clearly } x \in [0, 2)$$

$$\text{Consider } y = \frac{x^{3/2}}{\sqrt{2-x}}$$

When $x = 0$, $y = 0$

Also, when x increases from '0' to '2', y increases from '0' to ' ∞ '

Hence graph of given relation is as shown in the following figure:



Required Area is $A = 2 \int_0^2 \frac{x^{3/2}}{\sqrt{2-x}} dx$

Put $x = 2\sin^2 \theta$

$$\begin{aligned} \therefore A &= 2 \int_0^{\pi/2} \frac{2\sqrt{2} \sin^3 \theta}{\sqrt{2-2\sin^2 \theta}} 4\sin \theta \cos \theta d\theta \\ &= 16 \int_0^{\pi/2} \sin^4 \theta d\theta \\ &= 16 \int_0^{\pi/2} \frac{(1-\cos 2\theta)^2}{4} d\theta \\ &= 4 \int_0^{\pi/2} (1-2\cos 2\theta + \cos^2 2\theta) d\theta \\ &= 4 \int_0^{\pi/2} \left(1-2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta \\ &= 4 \left[\theta + \sin 2\theta + \frac{\theta + \frac{\sin 4\theta}{4}}{2} \right]_0^{\pi/2} \\ &= 4 \left(\frac{\pi}{2} + 0 + \frac{\pi}{4} \right) \\ &= 3\pi \end{aligned}$$

28. (4) $f(x) = x^2 + \cos x$
 $\Rightarrow f'(x) = 2x - \sin x$
 $\Rightarrow f'\left(\frac{\pi}{2}\right) = \pi - 1$

$\Rightarrow$ Equation of normal at $x = \frac{\pi}{2}$ is

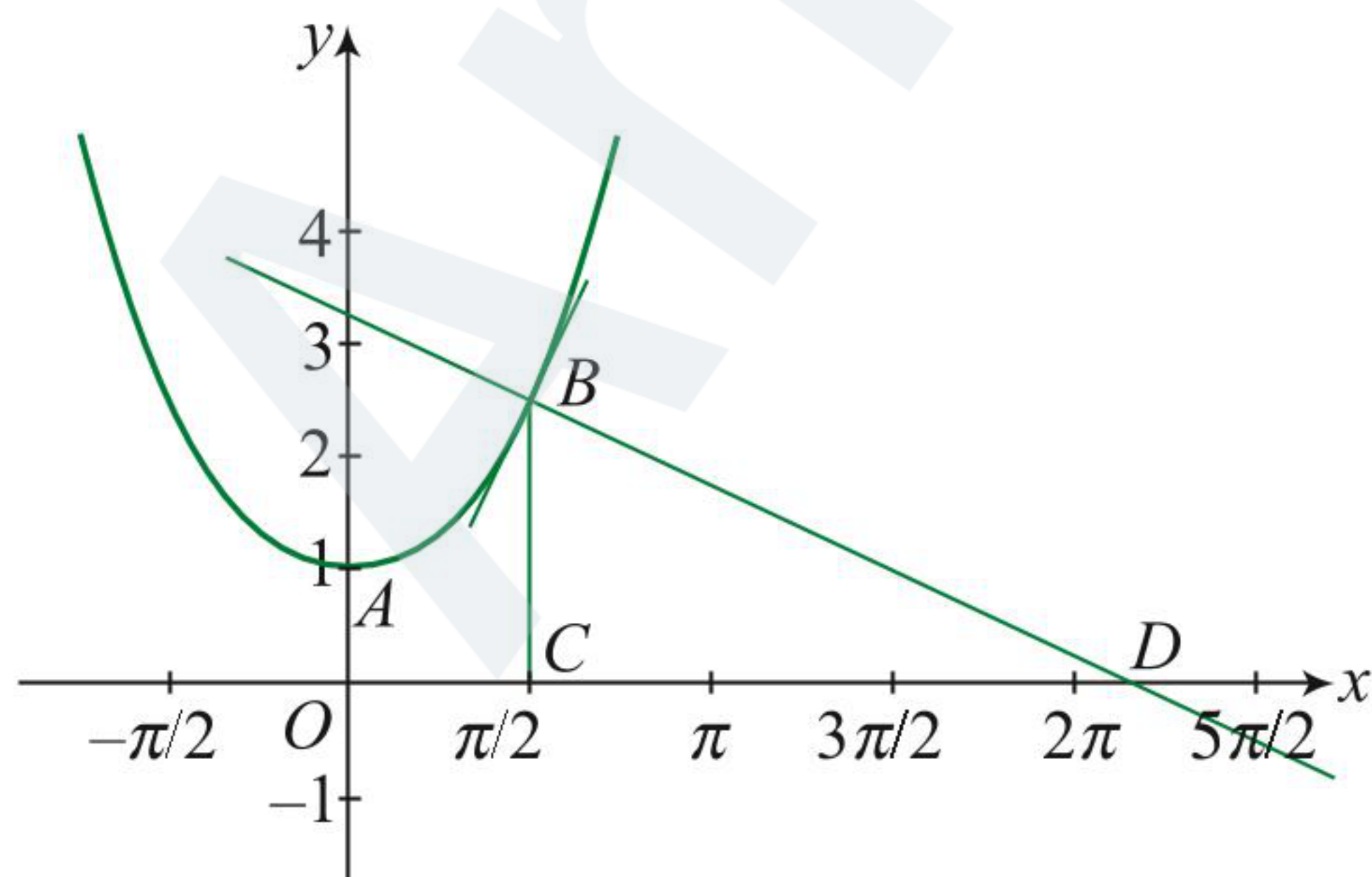
$$\left(y - \frac{\pi^2}{4}\right) = \frac{1}{1-\pi} \left(x - \frac{\pi}{2}\right)$$

It meets x -axis at $x = \frac{(\pi-1)\pi^2}{4} + \frac{\pi}{2}$

Also $f'(x) = 2x - \sin x > 0$ for $x > 0$ and < 0 for $x < 0$

So $f(x)$ increases for $x \in (0, \infty)$ and decreases for $x \in (-\infty, 0)$

Graph of the function is as shown in the following figure.



Required area

= Area $OABCO$ + Area of ΔBCD

$$= \int_0^{\pi/2} (x^2 + \cos x) dx + \frac{1}{2} \left[\frac{(\pi-1)\pi^2}{4} + \frac{\pi}{2} - \frac{\pi}{2} \right] \times \frac{\pi^2}{4}$$

$$\begin{aligned} &= \left[\frac{x^3}{3} + \sin x \right]_0^{\pi/2} + \frac{(\pi-1)\pi^2}{8} \\ &= \frac{\pi^3}{24} + 1 + \frac{\pi^4}{32} (\pi-1) \\ &= \frac{\pi^5}{32} - \frac{\pi^4}{32} + \frac{\pi^3}{24} + 1 \end{aligned}$$

29. (2) $\int_0^x e^t (\log_e \sec t - \sec^2 t) dt$

$$= \int_0^x e^t [(\log_e \sec t - \tan t) + (\tan t - \sec^2 t)] dt$$

$$= \left[e^t (\log_e \sec t - \tan t) \right]_0^x$$

$$= e^x (\log_e \sec x - \tan x)$$

$\therefore$ Required area,

$$A = \int_0^{\pi/3} [e^x (\log_e \sec x - \tan x) - (-2e^x \tan x)] dx$$

$$= \int_0^{\pi/3} e^x (\log_e \sec x + \tan x) dx$$

$$= \left[e^x \log_e \sec x \right]_0^{\pi/3}$$

$$= e^{\pi/3} \log_e \sec \frac{\pi}{3}$$

$$= e^{\pi/3} \log_e 2$$

30. (2) $ay^2 = x^2(a-x)$ or $y = \pm x \sqrt{\frac{a-x}{a}}$

Curve tracing : $y = x \sqrt{\frac{a-x}{a}}$

We must have $x \leq a$

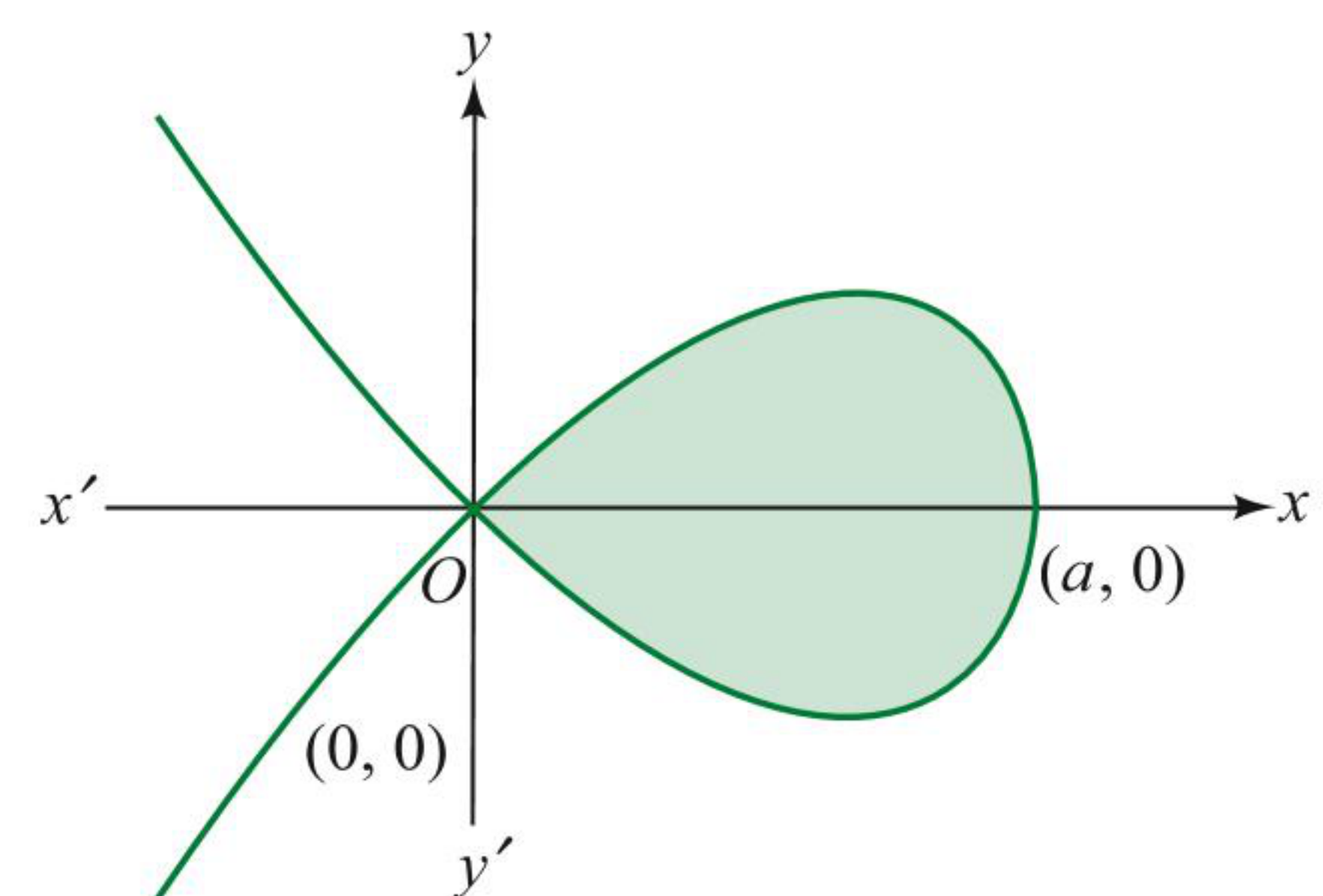
For $0 < x \leq a$, $y > 0$ and for $x < 0$, $y < 0$

Also $y = 0 \Rightarrow x = 0, a$

Curve is symmetrical about x -axis.

When $x \rightarrow -\infty$, $y \rightarrow -\infty$

Also, it can be verified that y has only one point of maxima for $0 < x < a$.



$$\text{Area} = 2 \int_0^a x \sqrt{\frac{a-x}{a}} dx$$

$$\sqrt{\frac{a-x}{a}} = t$$

$$\Rightarrow 1 - \frac{x}{a} = t^2 \text{ or } x = a(1 - t^2)$$

$$\Rightarrow A = 2 \int_1^0 a(1 - t^2)t(-2at)dt$$

$$= 4a^2 \int_0^1 (t^2 - t^4)dt$$

$$= 4a^2 \left[\frac{t^3}{3} - \frac{t^5}{5} \right]_0^1$$

$$= 4a^2 \left[\frac{1}{3} - \frac{1}{5} \right]$$

$$= \frac{8a^2}{15} \text{ sq. units.}$$

31. (4) $x = y^2 - 1$ is a parabola having vertex at $(-1, 0)$ and concave right hand side.

$$x = |y|\sqrt{1 - y^2}$$

Here $x \geq 0$ as R.H.S. is positive

When $y > 0$,

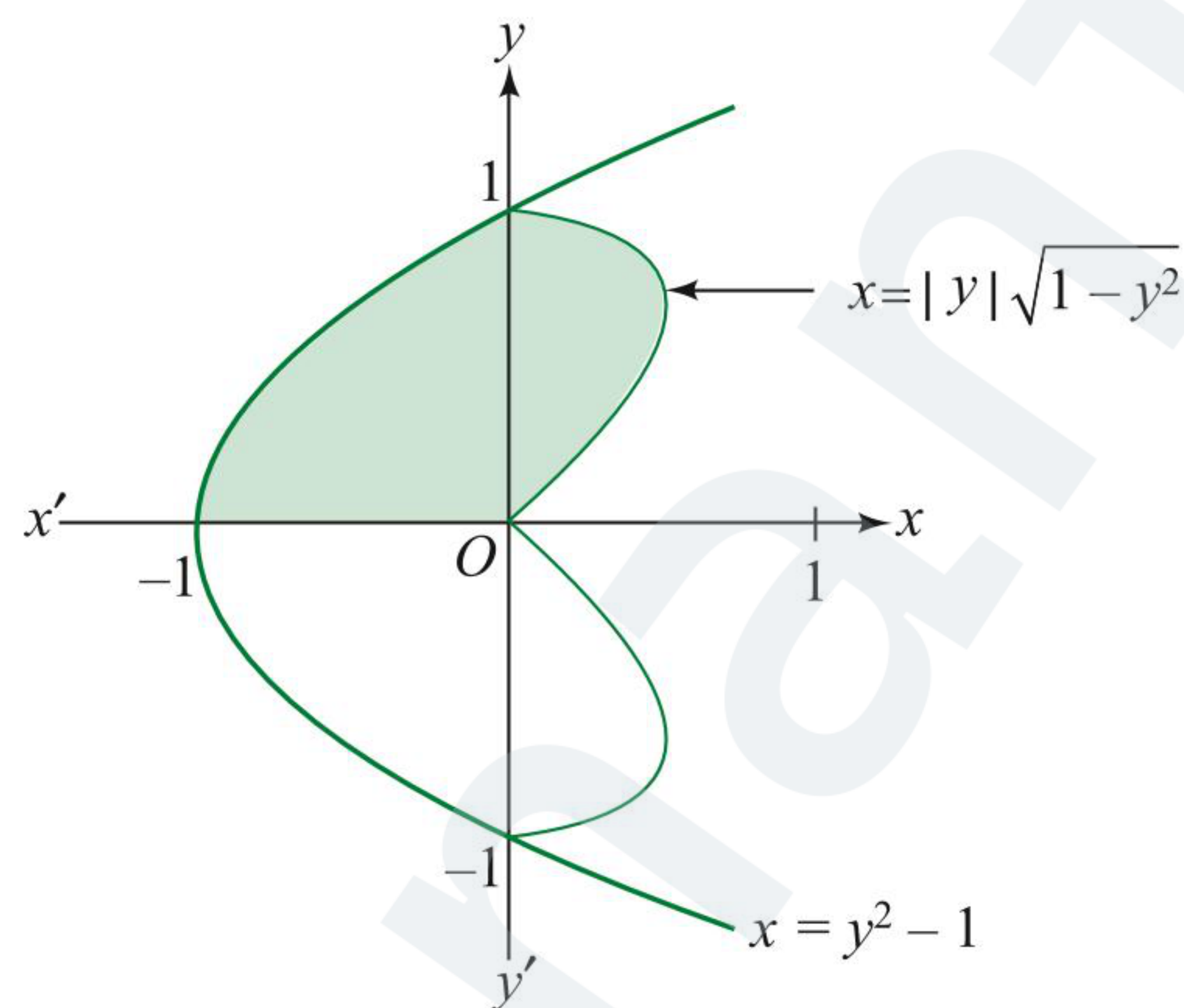
$$x = y\sqrt{1 - y^2} \quad (1)$$

(1) meets y -axis at $(0, 0)$, $(0, 1)$; thus,

$$\frac{dx}{dy} = \sqrt{1 - y^2} - \frac{y^2}{\sqrt{1 - y^2}} = \frac{1 - 2y^2}{\sqrt{1 - y^2}}$$

For $\frac{dx}{dy} = 0$, $y = \frac{1}{\sqrt{2}}$

Hence, graph of (1) is as shown in the figure, with $x = -y\sqrt{1 - y^2}$ ($y < 0$) as its mirror image.



$$A = 2 \int_0^1 [y\sqrt{1 - y^2} - (y^2 - 1)] dy$$

$$= 2 \text{ sq. units}$$

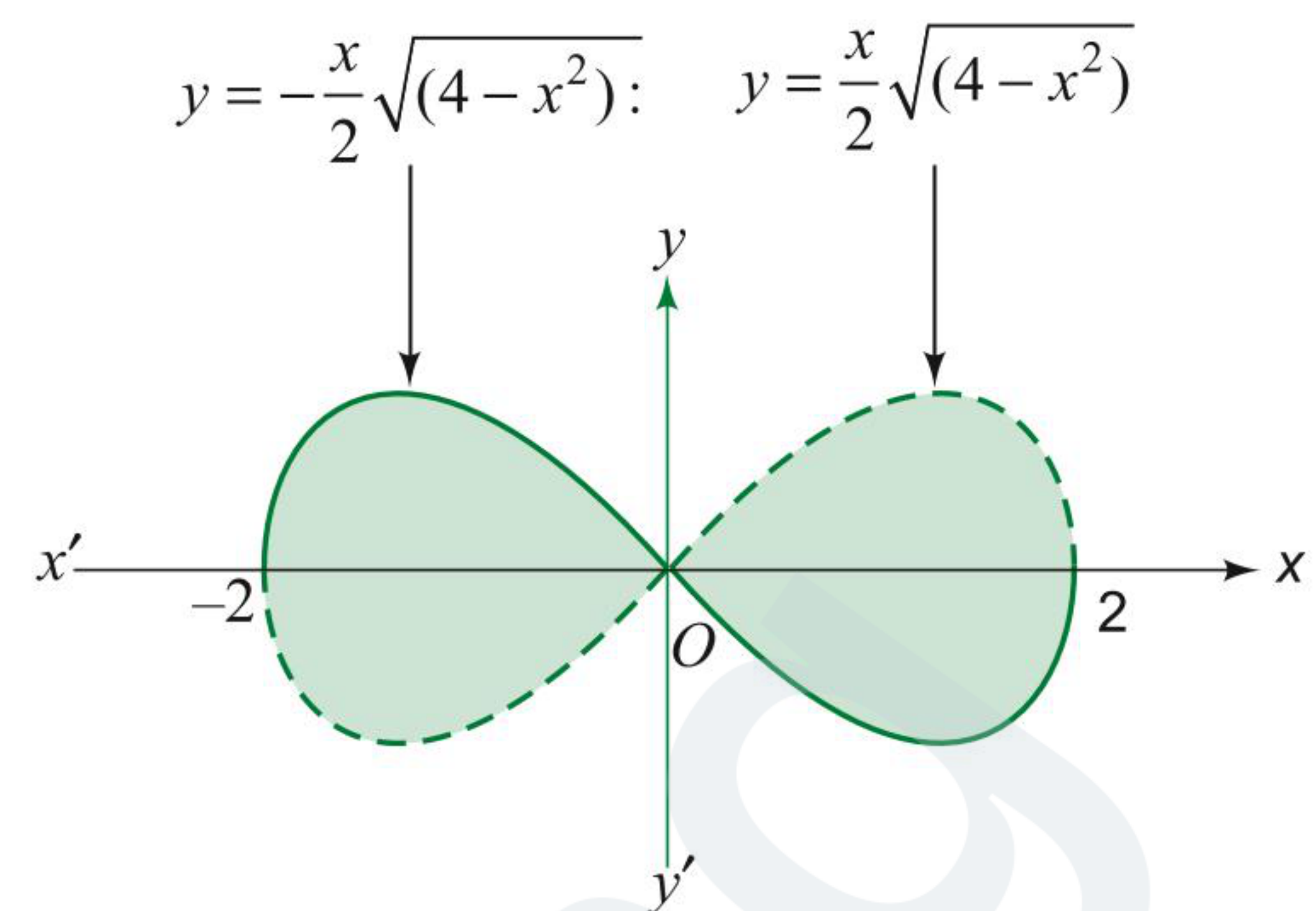
32. (4) $4y^2 = x^2(4 - x^2)$ (1)

$$\text{or } y = \pm \frac{1}{2} \sqrt{x^2(4 - x^2)}$$

$$= \pm \frac{x}{2} \sqrt{4 - x^2}$$

$$\Rightarrow y = -\frac{x}{2} \sqrt{4 - x^2};$$

$$y = \frac{x}{2} \sqrt{4 - x^2}$$



$$\therefore \text{Area } (A) = 4 \times \int_0^2 \frac{x}{2} \sqrt{4 - x^2} dx$$

$$\text{Let } 4 - x^2 = t \Rightarrow -2x dx = dt$$

$$\Rightarrow A = \int_0^4 \sqrt{t} dt = \left[\frac{t^{3/2}}{3/2} \right]_0^4 = \frac{2}{3} \times [\sqrt{64} - 0]$$

$$= \frac{16}{3} \text{ sq. units}$$

33. (1) The two curves are

$$xy^2 = a^2(a - x) \text{ or } x = \frac{a^3}{a^2 + y^2} \quad (1)$$

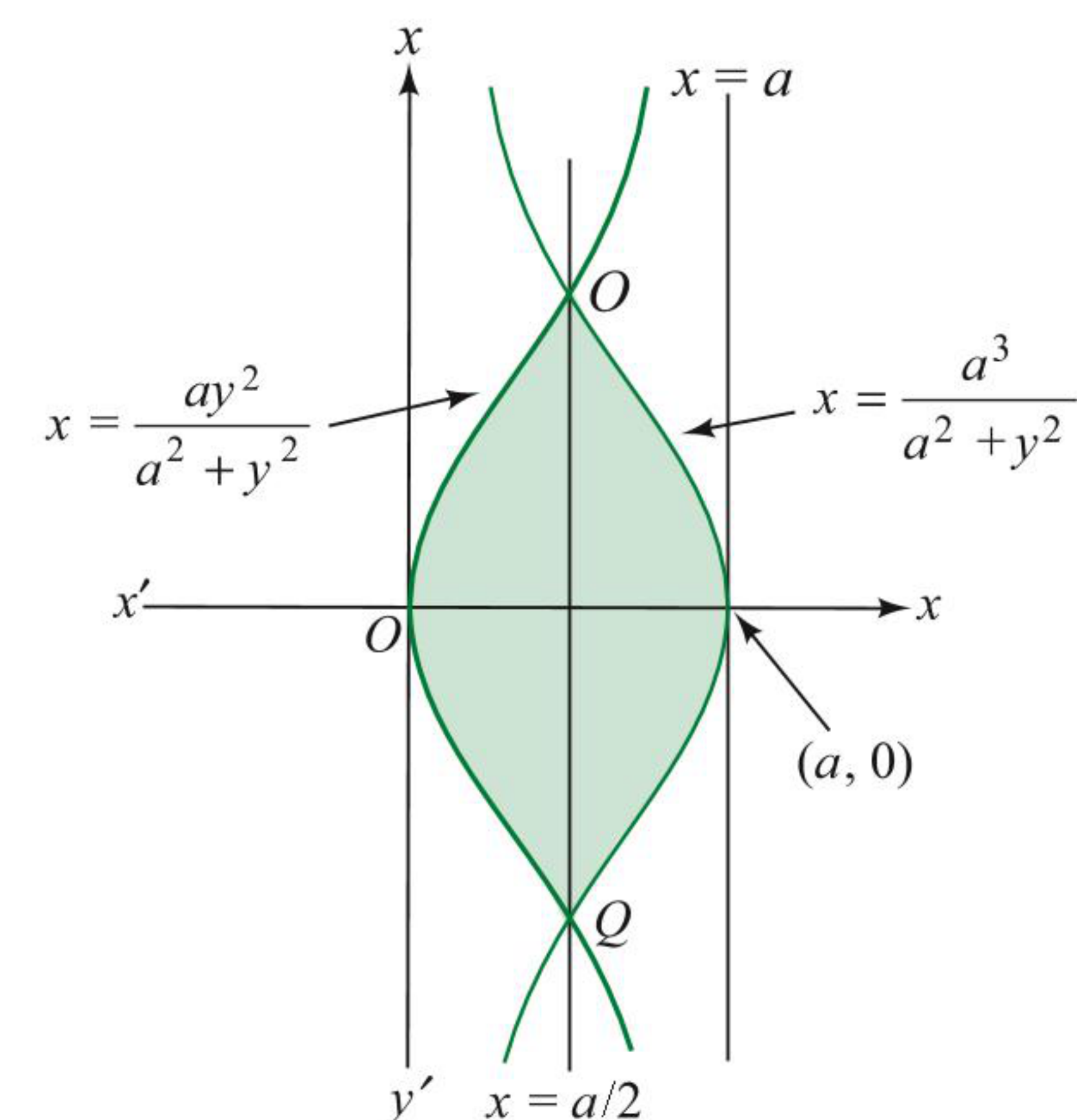
and $(a - x)y^2 = a^2x$

$$\Rightarrow x = \frac{ay^2}{a^2 + y^2} = \frac{ay^2 + a^3 - a^3}{a^2 + y^2} = a - \frac{a^3}{a^2 + y^2} \quad (2)$$

Curve (1) is symmetrical about x -axis and have y -axis as the asymptote.

Curve (2) is symmetrical about x -axis, tangent at origin as y -axis, and the asymptote $x = a$.

The two curves intersect at the point $P(a/2, a)$ and $Q(a/2, -a)$.



$$\text{Required area} = 2 \int_0^a \left[-a + \frac{a^3}{a^2 + y^2} + \frac{a^3}{a^2 + y^2} \right] dy$$

(integrating along y -axis)

$$= 2 \left[-ay + 2a^2 \tan^{-1} \frac{y}{a} \right]_0^a$$

$$= 2 \left[-a^2 + 2a^2 \frac{\pi}{4} \right]$$

$$= (\pi - 2)a^2 \text{ sq. units.}$$

34. (3) $(y - x)^2 = x^3$, where $x \geq 0 \Rightarrow y - x = \pm x^{3/2}$

$$\text{or } y = x + x^{3/2} \quad (1)$$

$$y = x - x^{3/2} \quad (2)$$

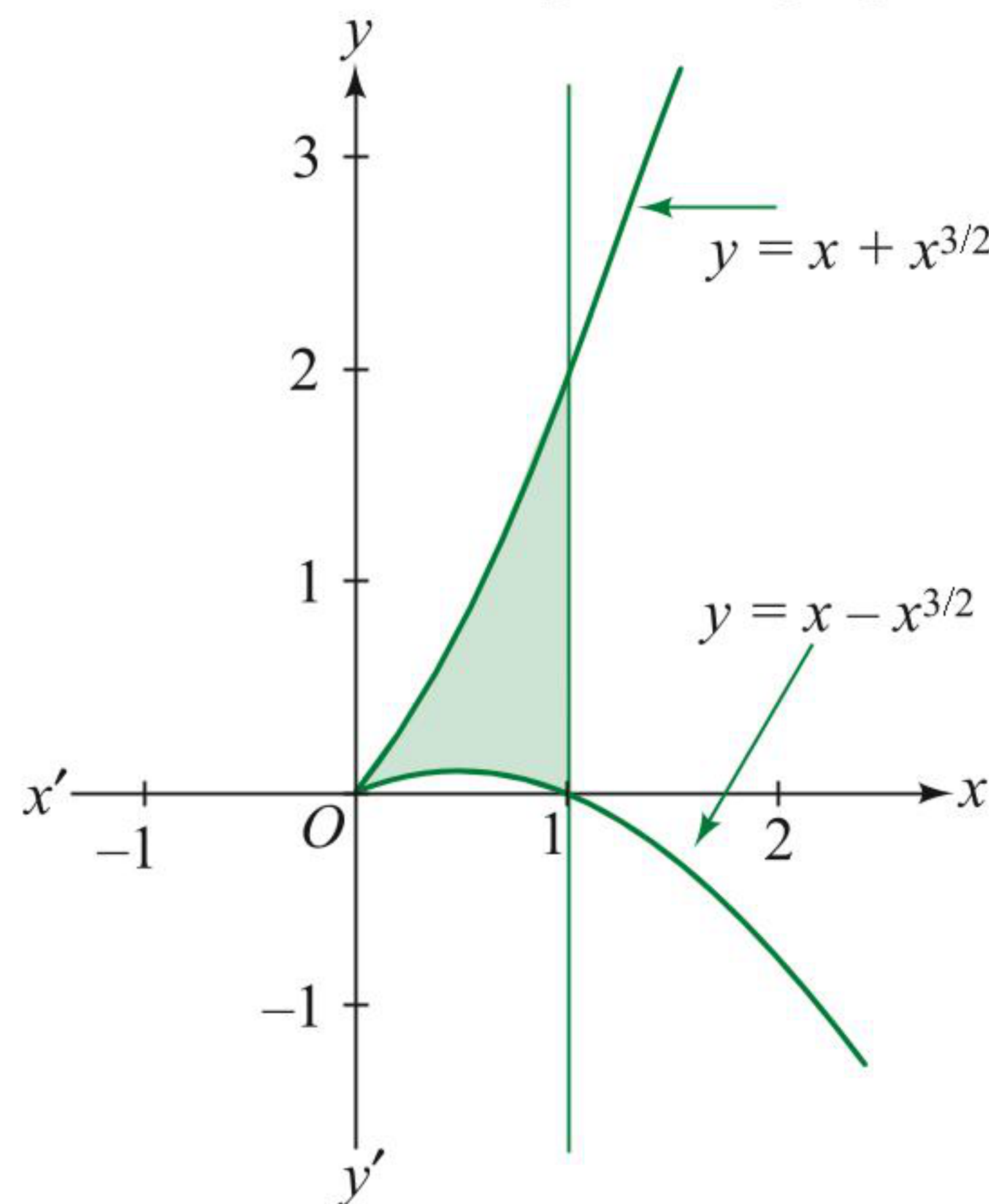
Function (1) is an increasing function.

Function (2) meets x -axis, when $x - x^{3/2} = 0$ or $x = 0, 1$.

Also, for $0 < x < 1$, $x - x^{3/2} > 0$ and for $x > 1$, $x - x^{3/2} < 0$.

When $x \rightarrow \infty$, $x - x^{3/2} \rightarrow -\infty$.

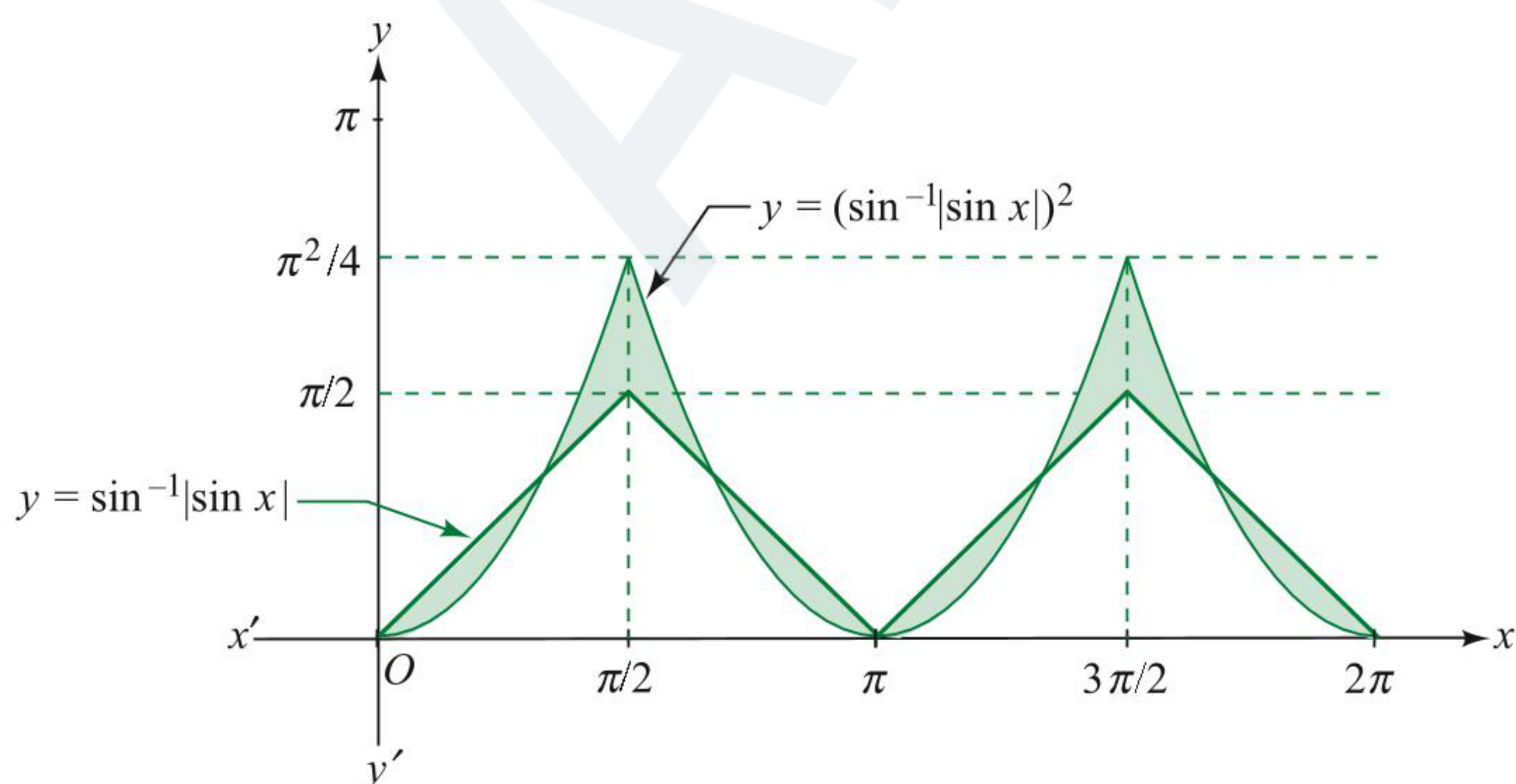
From these information, we can plot the graph as shown.



$$\begin{aligned} \text{Required area} &= \int_0^1 \left[(x + x^{3/2}) - (x - x^{3/2}) \right] dx \\ &= 2 \int_0^1 x^{3/2} dx \\ &= 2 \left[\frac{x^{5/2}}{5/2} \right]_0^1 = \frac{4}{5} \text{ sq. units.} \end{aligned}$$

$$\begin{aligned} 35. (4) \quad y = \sin^{-1} |\sin x| &= \begin{cases} x, & 0 \leq x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x < \pi \\ x - \pi, & \pi \leq x < \frac{3\pi}{2} \\ 2\pi - x, & \frac{3\pi}{2} \leq x < 2\pi \end{cases} \\ y = (\sin^{-1} |\sin x|)^2 &= \begin{cases} x^2, & 0 \leq x < \frac{\pi}{2} \\ (\pi - x)^2, & \frac{\pi}{2} \leq x < \pi \\ (x - \pi)^2, & \pi \leq x < \frac{3\pi}{2} \\ (2\pi - x)^2, & \frac{3\pi}{2} \leq x < 2\pi. \end{cases} \end{aligned}$$

The required area A is shown as shaded region in the figure.



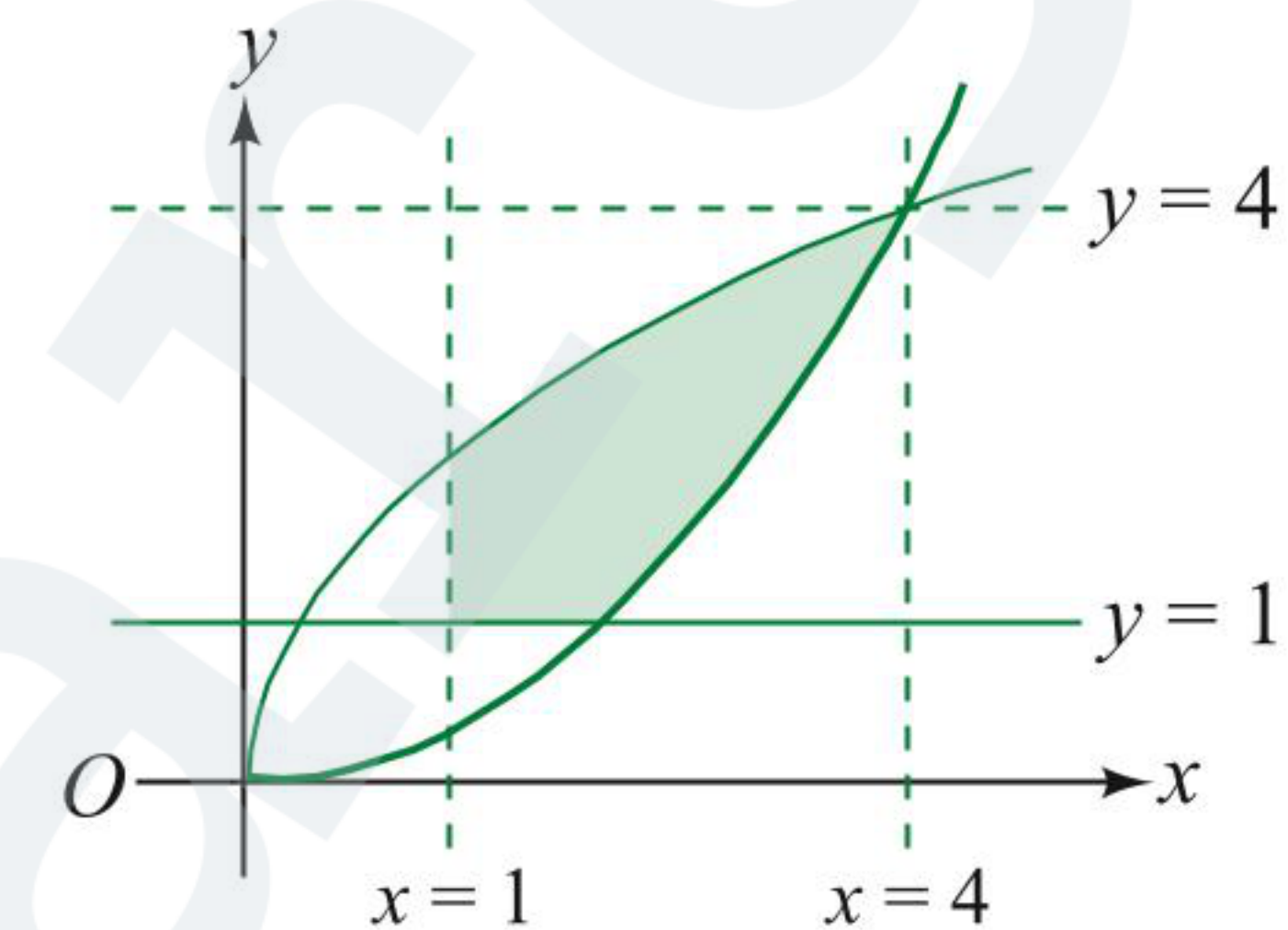
$$\begin{aligned} \Rightarrow & 4 \int_0^1 (x - x^2) dx + 4 \int_1^{\pi/2} (x^2 - x) dx \\ &= \frac{4}{3} + \pi^2 \left[\frac{\pi - 3}{6} \right] \text{ sq. units.} \end{aligned}$$

$$36. (3) \quad y^2 = 4[\sqrt{y}]x$$

For $y \in [1, 4)$, $[\sqrt{y}] = 1$ or $y^2 = 4x$.

Similarly, for $x \in [1, 4)$, $[\sqrt{x}] = 1$ and

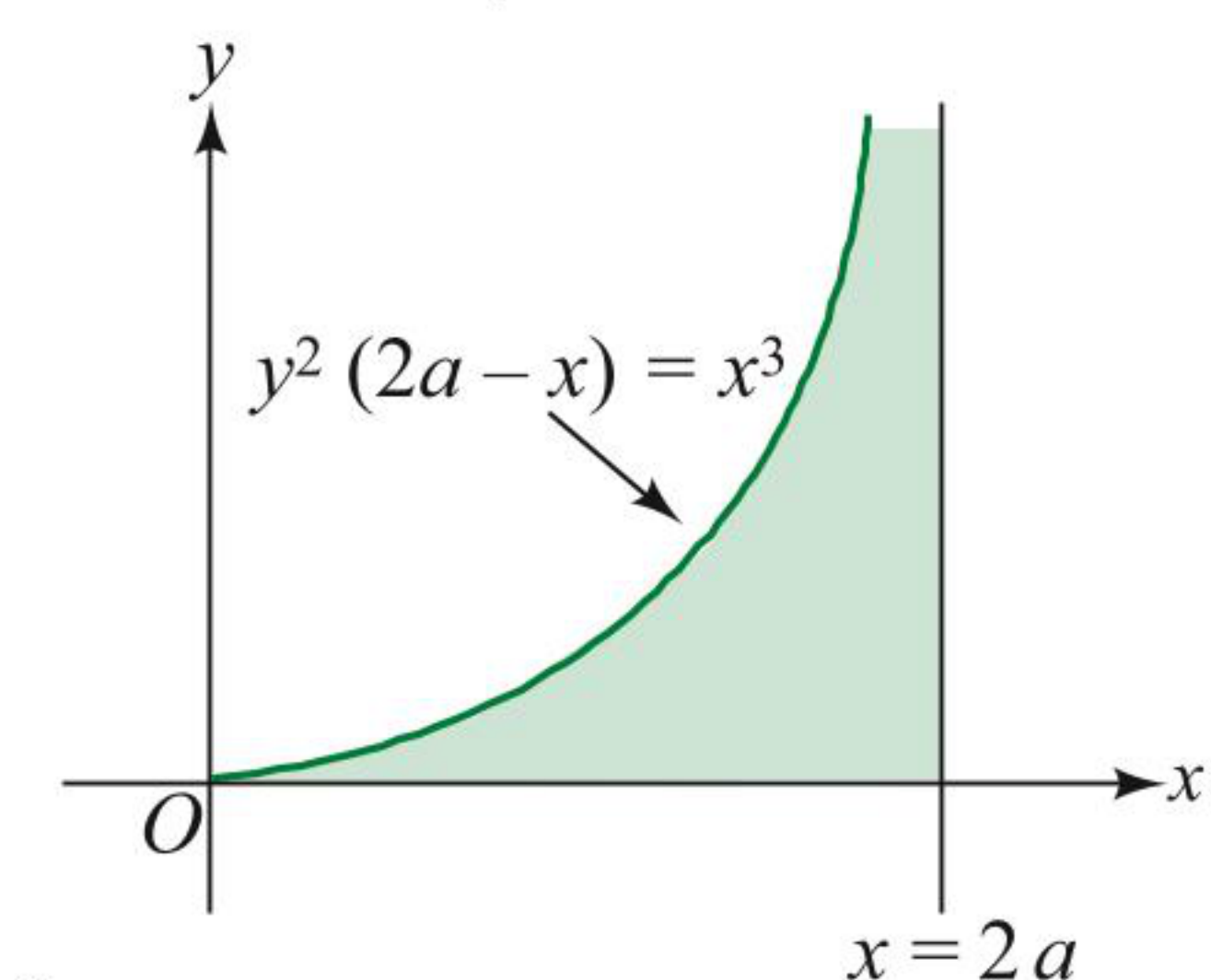
$x^2 = 4[\sqrt{x}]y$ would transform into $x^2 = 4y$.



The required area is the shaded region.

$$\begin{aligned} A &= \int_1^2 (2\sqrt{x} - 1) dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx \\ &= \left(\frac{4}{3} x^{3/2} - x \right)_1^2 + \left(\frac{4}{3} x^{3/2} - \frac{x^3}{12} \right)_2^4 = \frac{11}{3} \text{ sq. units.} \end{aligned}$$

$$37. (2) \quad \text{The required area } A = \int_0^{2a} \sqrt{\frac{x^3}{2a-x}} dx$$



Put $x = 2a \sin^2 \theta$

$$\Rightarrow dx = 2a \cdot 2 \sin \theta \cos \theta d\theta$$

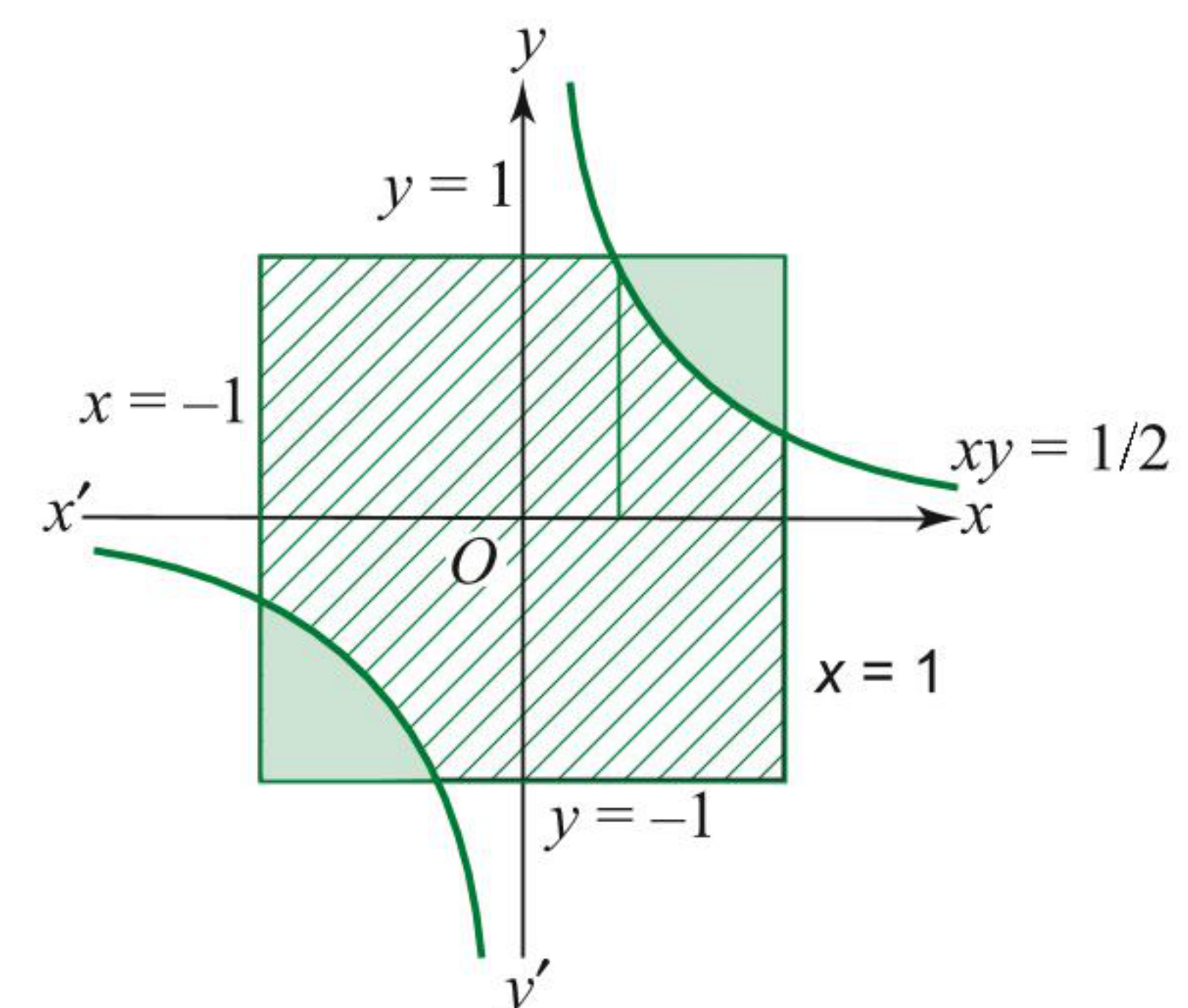
$$\Rightarrow A = 8a^2 \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - 2\cos 2\theta + \cos^2 2\theta) d\theta$$

$$= 2a^2 \int_0^{\pi} \left(1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) d\theta = \frac{3\pi a^2}{2}$$

$$38. (2) \quad \max(|x|, |y|) \leq 1 \Rightarrow |x| \leq 1 \text{ and } |y| \leq 1$$

which represent square bounded by $x = \pm 1$ and $y = \pm 1$



The required area is the lined area.

Now, shaded area is given by

$$\begin{aligned} 2 \int_{1/2}^1 \left(1 - \frac{1}{2x}\right) dx &= 2 \left(x - \frac{1}{2} \ln x \right)_{1/2}^1 \\ &= 2 \left[(1-0) - \left(\frac{1}{2} - \frac{1}{2} \ln \frac{1}{2} \right) \right] \\ &= 1 - \ln 2 \text{ sq. units.} \end{aligned}$$

or Horizontal lined area = $4 - (1 - \ln 2) = 3 + \ln 2$ sq. units.

39. (1) The points in the required region satisfy

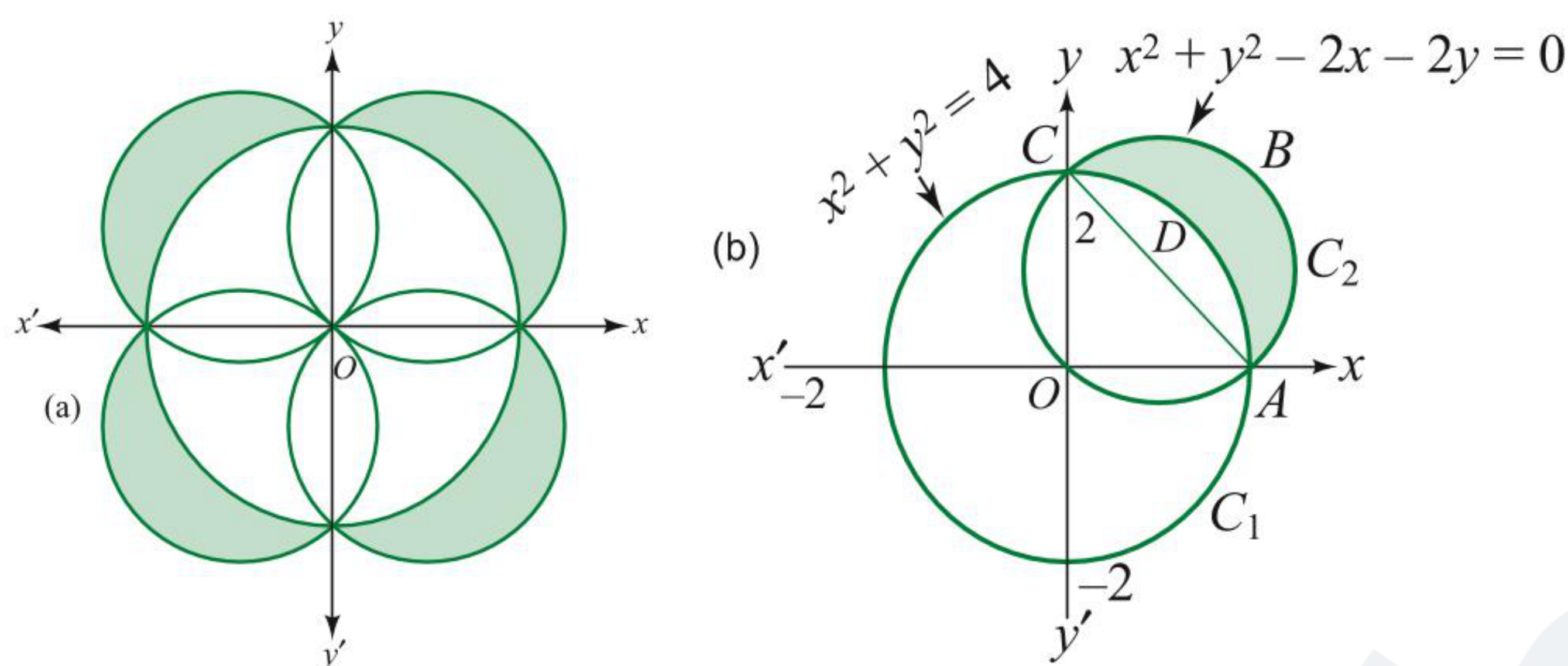
$$4 \leq x^2 + y^2 \leq 2(|x| + |y|) \quad \dots(1)$$

Since the curve (1) is symmetrical about both the axes, the required area is 4 times the area of the region in the first quadrant. Therefore, it is sufficient to sketch the region and to find the area in the first quadrant.

In the first quadrant, the curve (1) consist of two curves

$$x^2 + y^2 \geq 4 \quad (C_1)$$

$$\text{and } x^2 + y^2 - 2x - 2y \geq 0 \quad (C_2)$$



$$\begin{aligned} \therefore \text{ Required area} &= 4 (\text{area } ABCDA) \\ &= 4(\text{area of semi-circle } ABCA) - (\text{area of sector } ADCA) \\ &= 4(\text{area of semi-circle } ABCA) - (\text{area of sector } OADCO \\ &\quad - \text{area of triangle } OAC) \\ &= 4\{\pi - (\pi - 2)\} = 8 \text{ sq. units.} \end{aligned}$$

$$40. (3) \int_{\pi/4}^{\beta} f(x) dx = \beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2} \beta$$

Differentiating both sides w.r.t. β , we get

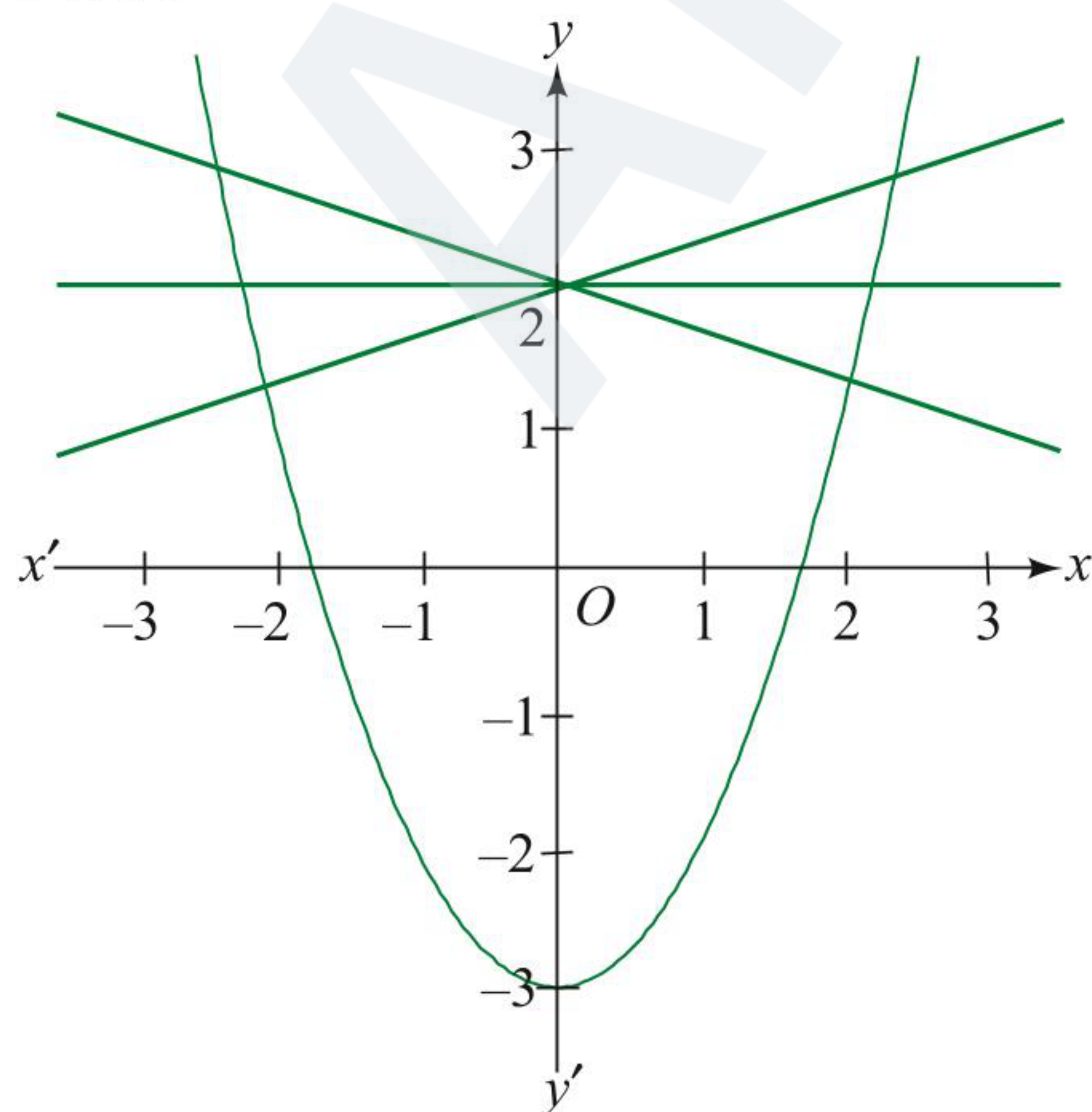
$$\therefore f(\beta) = \beta \cos \beta + \sin \beta - \frac{\pi}{4} \sin \beta + \sqrt{2}$$

$$\Rightarrow f'(\beta) = -\beta \sin \beta + \cos \beta + \cos \beta - \frac{\pi}{4} \cos \beta$$

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

Multiple Correct Answers Type

1. (2), (3)



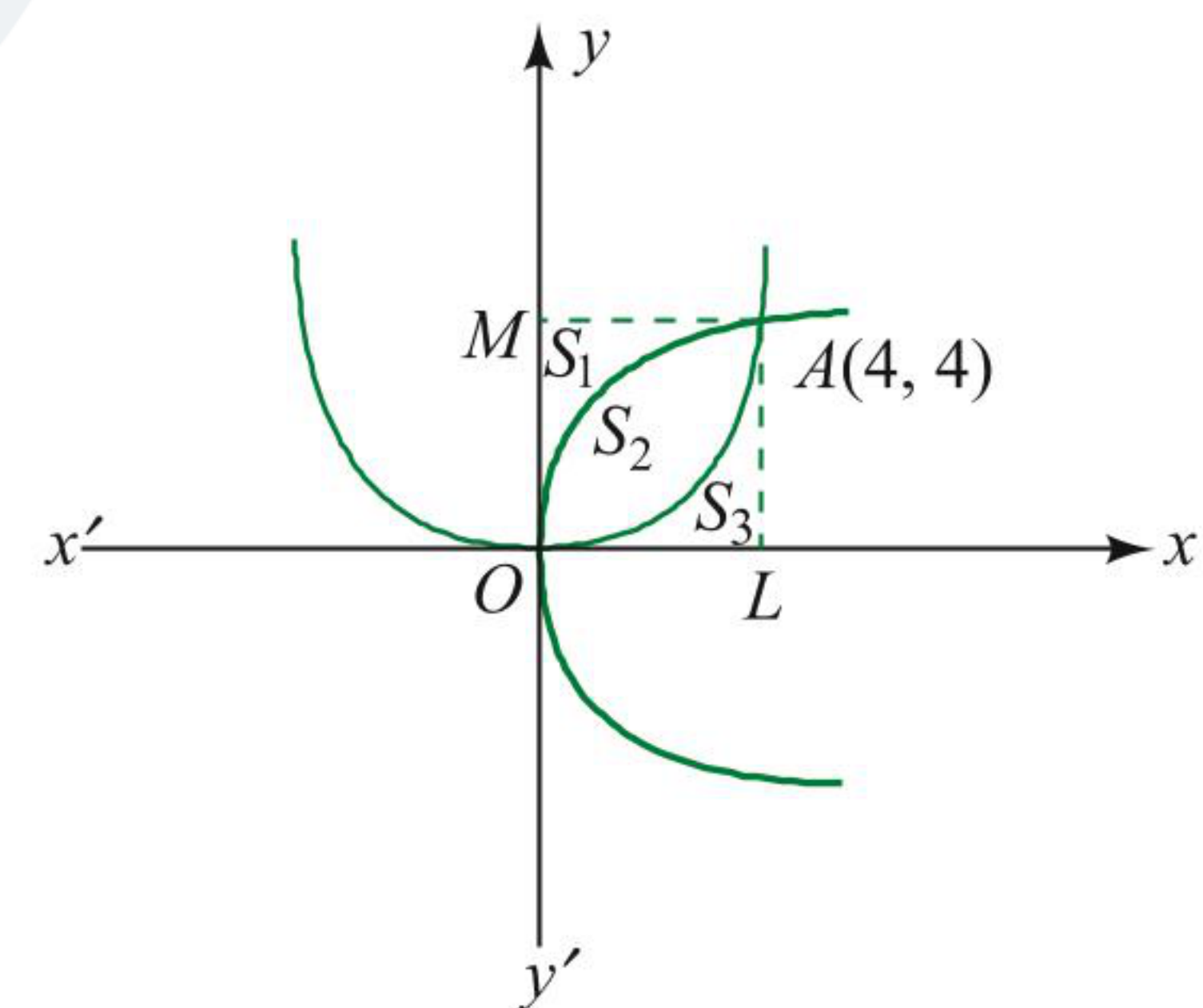
Line $y = kx + 2$ passes through fixed point $(0, 2)$ for different value of k .

Also, it is obvious that minimum $A(k)$ occurs when $k = 0$, as when line is rotated from this position about point $(0, 2)$, the increased part of area is more than the decreased part of area. Therefore,

$$\begin{aligned} \text{Minimum area} &= 2 \int_0^{\sqrt{5}} (2 - (x^2 - 3)) dx \\ &= 2 \int_0^{\sqrt{5}} (5 - x^2) dx \\ &= 2 \left[5x - \frac{x^3}{3} \right]_0^{\sqrt{5}} \\ &= 2 \left[5\sqrt{5} - \frac{5\sqrt{5}}{3} \right] \\ &= \frac{20\sqrt{5}}{3} \text{ sq. units} \end{aligned}$$

2. (1), (3), (4)

$y^2 = 4x$ and $x^2 = 4y$ meet at $O(0, 0)$ and $A(4, 4)$.



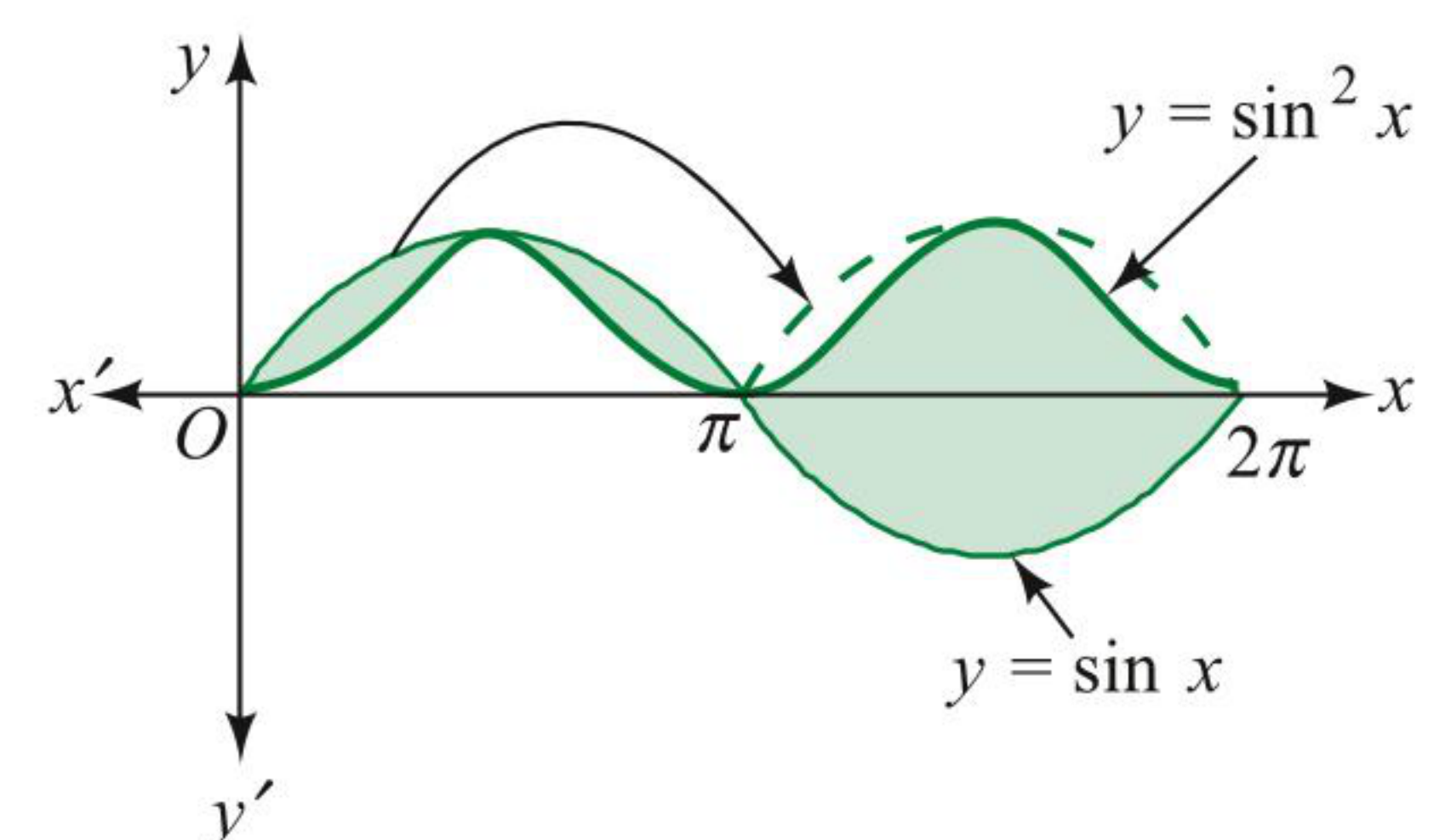
$$\text{Now } S_3 = \int_0^4 \frac{x^2}{4} dx = \frac{1}{4} \left[\frac{x^3}{3} \right]_0^4 = \frac{1}{12} [64 - 0] = \frac{16}{3}.$$

$$\begin{aligned} S_2 &= \int_0^4 2\sqrt{x} dx - S_3 = 2 \left[\frac{x^{3/2}}{3/2} \right]_0^4 - \frac{16}{3} \\ &= \frac{4}{3} [8 - 0] - \frac{16}{3} = \frac{16}{3}. \end{aligned}$$

$$\text{And } S_1 = 4 \times 4 - (S_2 + S_3) = 16 - \left(\frac{16}{3} + \frac{16}{3} \right) = \frac{16}{3}.$$

Hence, $S_1:S_2:S_3 = 1:1:1$

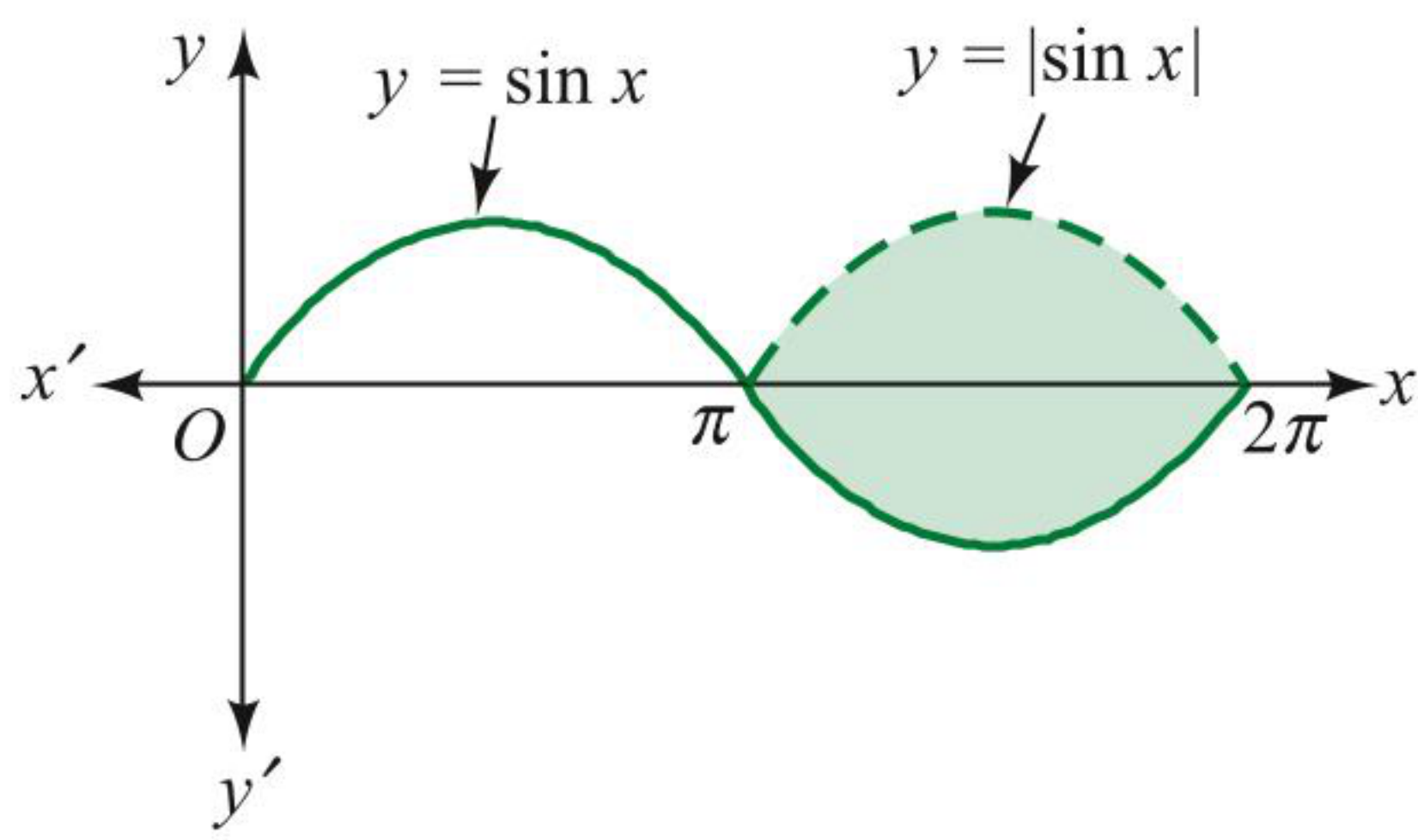
3. (1), (3), (4)



We know that area bounded by $y = \sin x$ and x -axis for $x \in [0, \pi]$ is 2 sq. units.

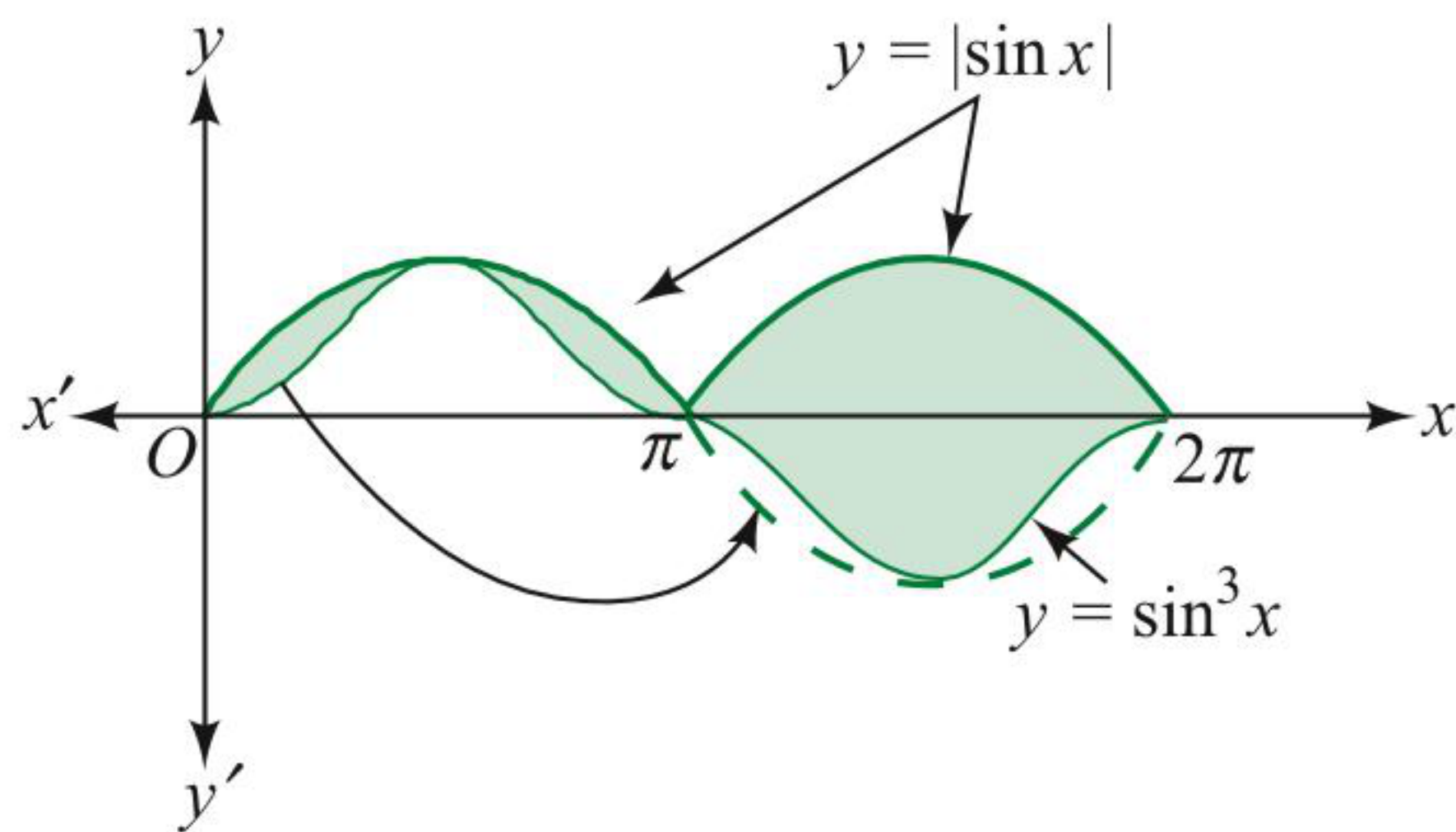
Then area bounded by $y = \sin x$ and $y = \sin^2 x$ is 4 sq. units for $x \in [0, 2\pi]$.

Then for $x \in [0, 10\pi]$, the area bounded is 20 sq. units.



The area bounded by $y = \sin x$ and $y = |\sin x|$ for $x \in [0, 2\pi]$ is 4 sq. units.

Then for $x \in [0, 20\pi]$, the area bounded is 40 sq. units.



The area bounded by $y = \sin x$ and $y = \sin^3 x$ for $x \in [0, 2\pi]$ is 4 sq. units.

Then for $x \in [0, 10\pi]$, the area bounded is 20 sq. units.

Similarly, the area bounded by $y = \sin x$ and $y = \sin^4 x$ for $x \in [0, 10\pi]$ is 20 sq. units.

4. (3), (4)

Since the curve $y = ax^{1/2} + bx$ passes through the point (1, 2)

$$\therefore 2 = a + b \quad (1)$$

By observation the curve also passes through (0, 0). Therefore, the area enclosed by the curve, x-axis and $x = 4$ is given by

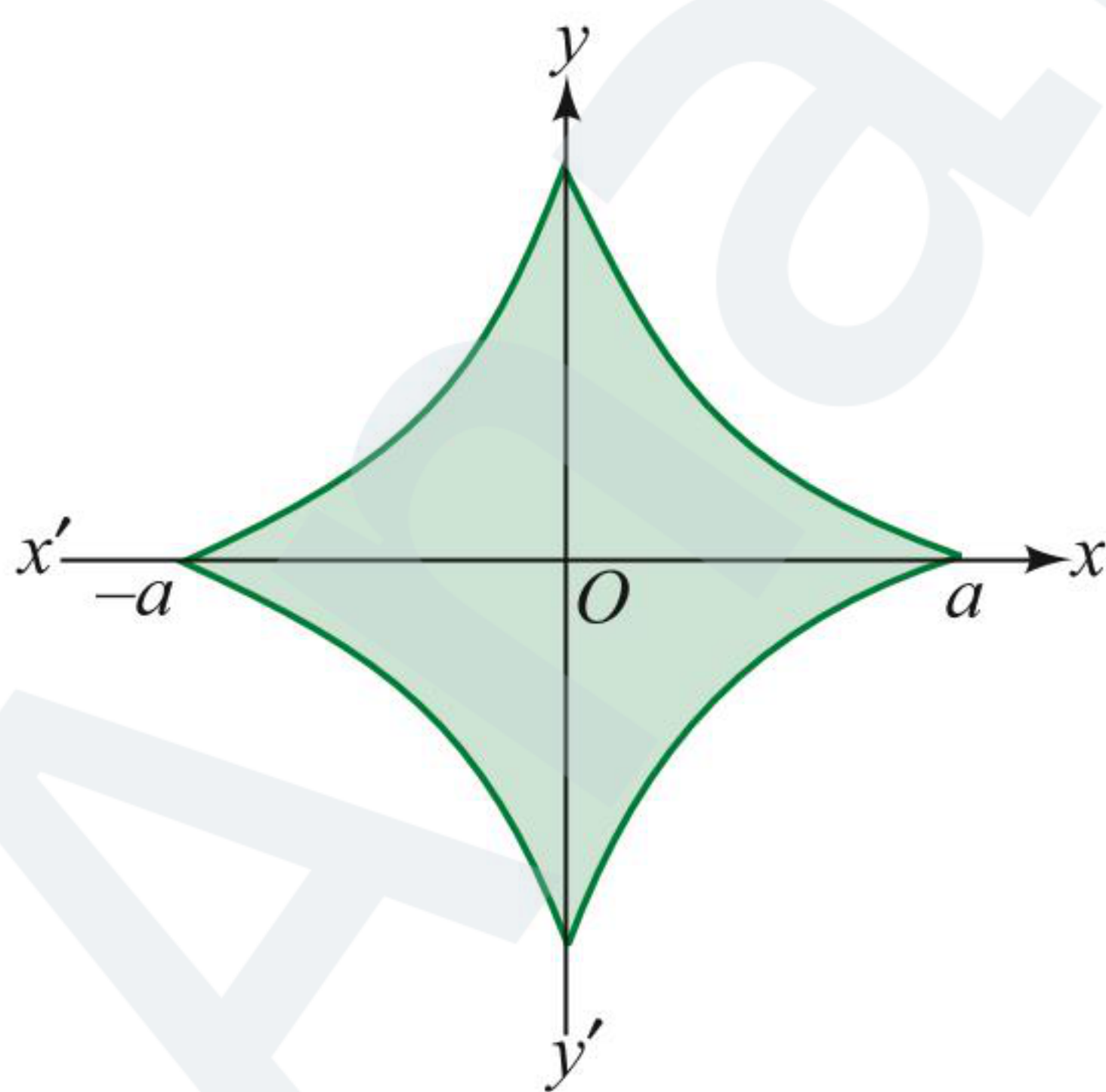
$$A = \int_0^4 (ax^{1/2} + bx) dx = 8 \text{ or } \frac{2a}{3} \times 8 + \frac{b}{2} \times 16 = 8$$

$$\text{or } \frac{2a}{3} + b = 1. \quad (2)$$

Solving (1) and (2), we get $a = 3$, $b = -1$.

5. (1), (3), (4)

Eliminating t , we have $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow y = (a^{2/3} - x^{2/3})^{3/2}$.



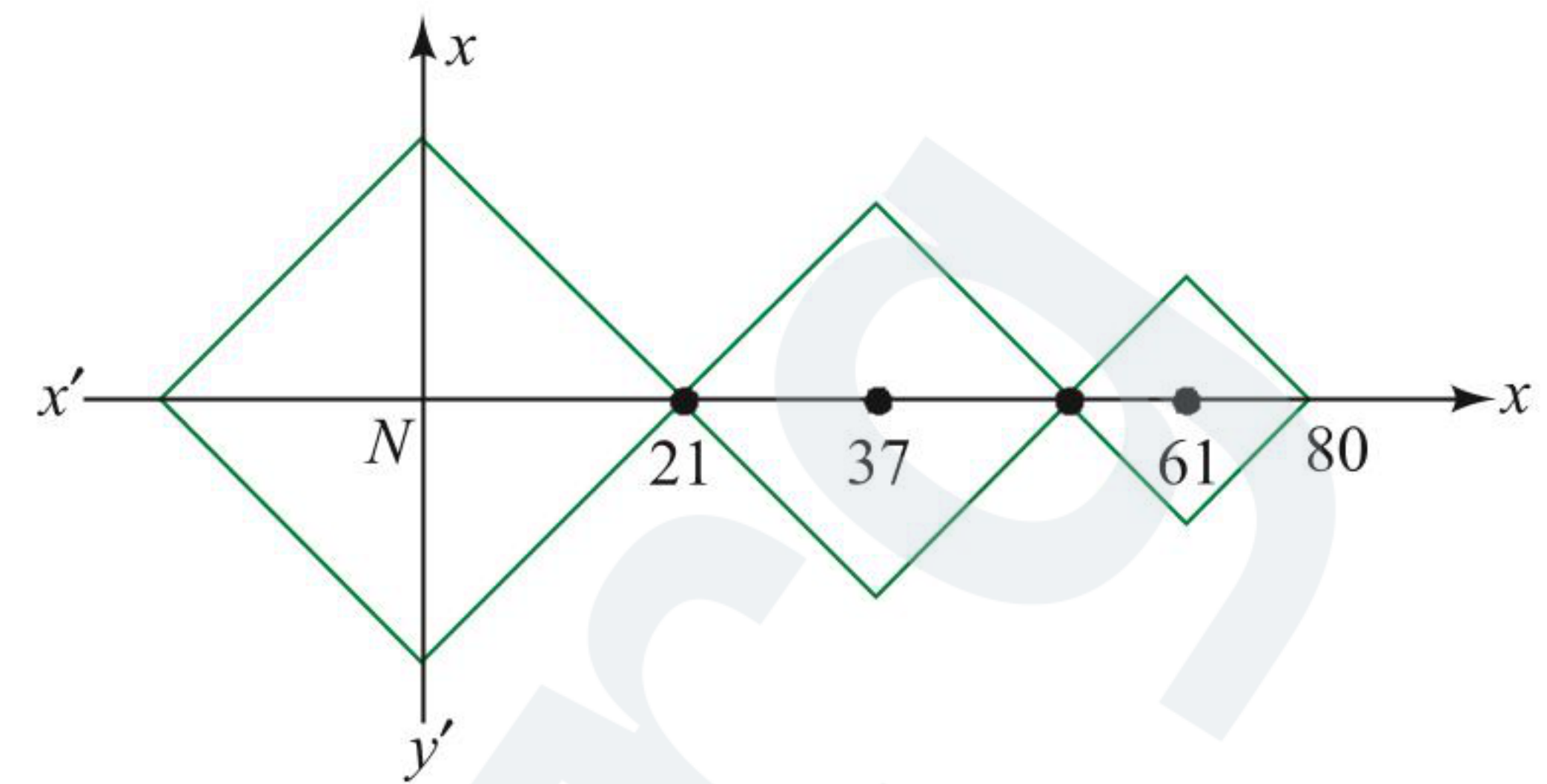
From the figure,

$$\begin{aligned} A &= 2 \int_{-a}^a (a^{2/3} - x^{2/3})^{3/2} dx \\ &= 4 \int_0^a (a^{2/3} - x^{2/3})^{3/2} dx = 4 \int_0^a y dx \\ &= 4a^2 \int_0^{\pi/2} 3 \cos^3 t \sin^2 t \cos t dt. \end{aligned}$$

6. (1), (3)

$$a_1 = 0, b_1 = 32, a_2 = a_1 + \frac{3}{2} b_1 = 48, b_2 = \frac{b_1}{2} = 16$$

$$a_3 = 48 + \frac{3}{2} \times 16 = 72, b_3 = \frac{16}{2} = 8$$



So the three loops from $i = 1$ to $i = 3$ are alike.

Now area of i th loop (square) = $\frac{1}{2}$ (diagonal)²

$$A_i = \frac{1}{2} (2b_i)^2 = 2(b_i)^2$$

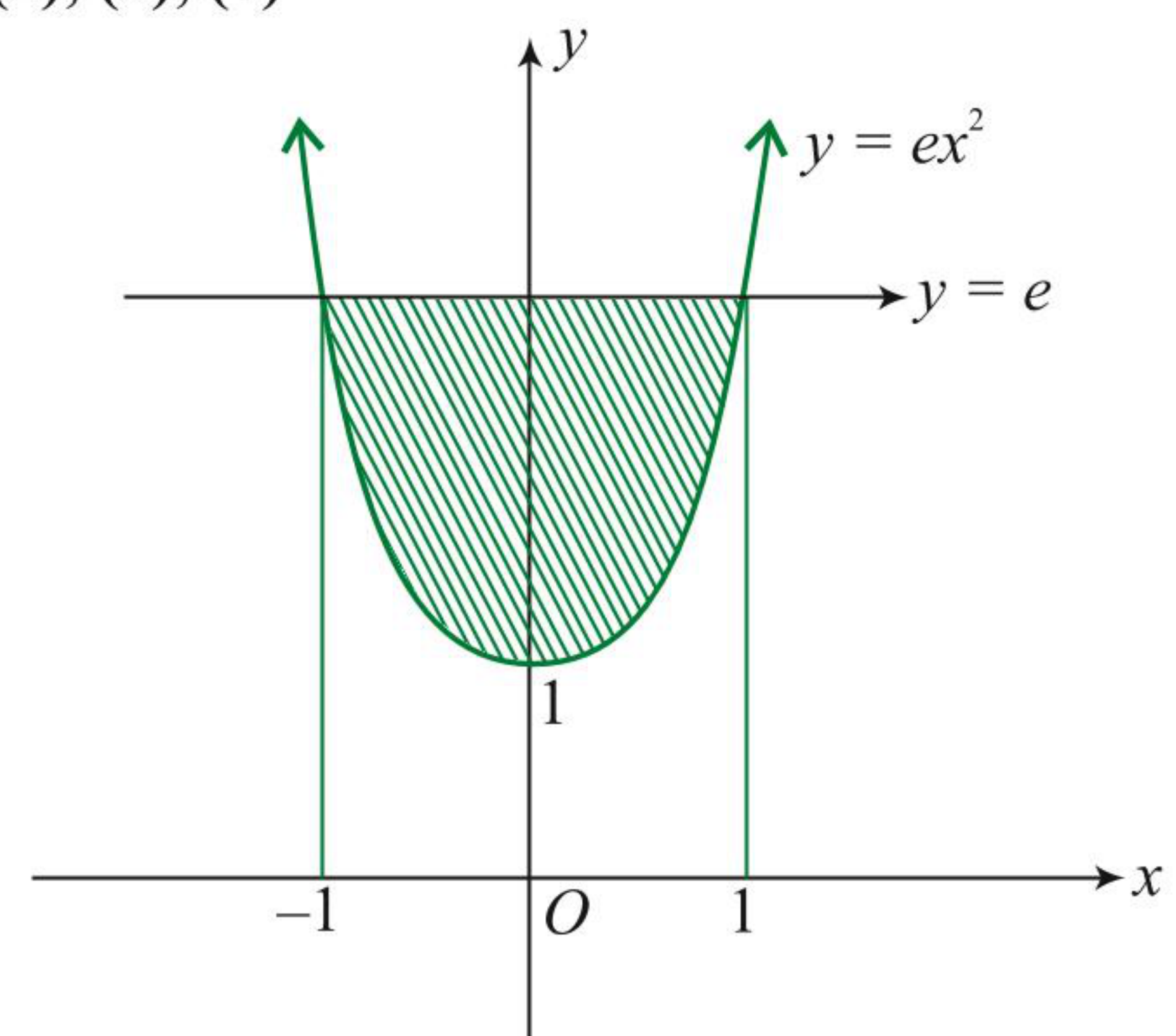
$$\text{So, } \frac{A_{i+1}}{A_i} = \frac{2(b_{i+1})^2}{2(b_i)^2} = \frac{1}{4}.$$

So the areas form a G.P. series.

So, the sum of the G.P. up to infinite terms is

$$\begin{aligned} A_1 \frac{1}{1-r} &= 2(32)^2 \times \frac{1}{1-\frac{1}{4}} \\ &= 2 \times (32)^2 \times \frac{4}{3} = \frac{8}{3} (32)^2 \text{ sq. units.} \end{aligned}$$

7. (1), (2), (3), (4)



$$\text{Area, } A = \int_{-1}^1 (e - e^{x^2}) dx = 2e - \int_{-1}^1 e^{x^2} dx$$

$$\begin{aligned} y &= e^{x^2} \\ \therefore \log_e y &= x^2 \\ \therefore x &= \pm \sqrt{\log_e y} \end{aligned}$$

$$\therefore \text{Area, } A = 2 \int_1^e \sqrt{\log_e y} dy$$

Now put $\log_e y = t$.

$$\begin{aligned} \therefore y &= e^t \\ \therefore dy &= e^t dt \end{aligned}$$

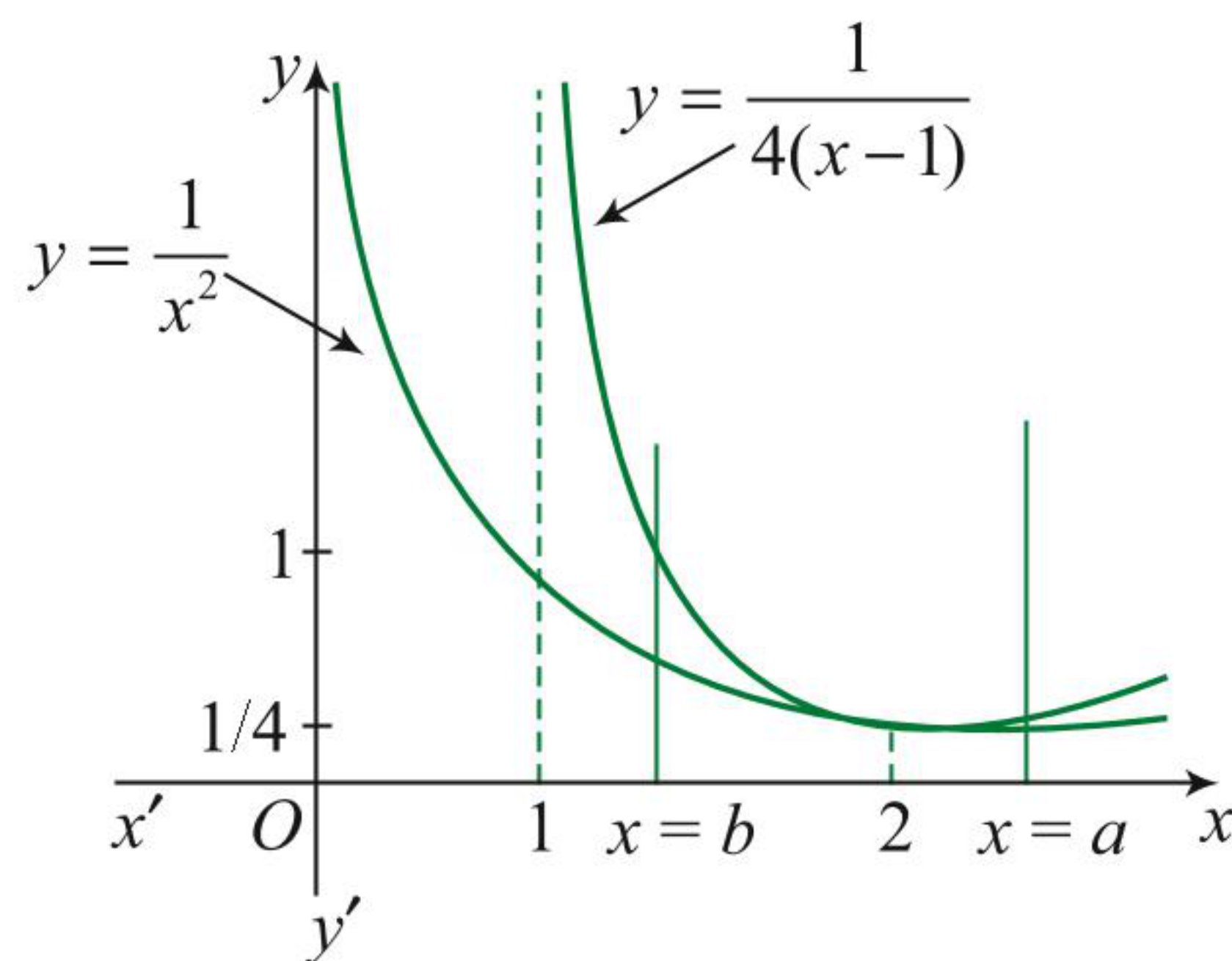
$$\text{So, area } A = 2 \int_0^1 \sqrt{t} e^t dt$$

8. (1), (4)

Solving given curves $y = \frac{1}{x^2}$; $y = \frac{1}{4(x-1)}$, we get

$$x^2 = 4(x-1) \Rightarrow (x-2)^2 = 0$$

So, curves touch other at $x = 2$.



$$\therefore A = \int_2^a \left(\frac{1}{4(x-1)} - \frac{1}{x^2} \right) dx = \frac{1}{a}$$

$$\Rightarrow \left[\frac{1}{4} \log(x-1) + \frac{1}{x} \right]_2^a = \frac{1}{a}$$

$$\Rightarrow \frac{1}{4} \log(a-1) + \frac{1}{a} - \frac{1}{2} = \frac{1}{a}$$

$$\Rightarrow \log(a-1) = 2$$

$$\Rightarrow a = e^2 + 1$$

$$\text{Also } 1 - \frac{1}{b} = \int_b^2 \left(\frac{1}{4(x-1)} - \frac{1}{x^2} \right) dx$$

$$\Rightarrow b = 1 + e^{-2}$$

9. (1), (2), (4)

$$\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$$

Consider $x, y > 0$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\therefore y = (\sqrt{a} - \sqrt{x})^2$$

$$y' = 2(\sqrt{a} - \sqrt{x}) \left(-\frac{1}{\sqrt{x}} \right) = 2 \left(1 - \frac{\sqrt{a}}{\sqrt{x}} \right)$$

$$\therefore y'' = \frac{\sqrt{a}}{x^{3/2}} > 0$$

So, graph is concave upward.

Area bounded by S_1 and coordinate axis in first quadrant

$$\begin{aligned} &= \int_0^a (\sqrt{a} - \sqrt{x})^2 dx \\ &= \int_0^a [a + x - 2\sqrt{a}\sqrt{x}] dx \\ &= \left[ax + \frac{x^2}{2} - 4\sqrt{a} \frac{x^{3/2}}{3} \right]_0^a \\ &= a^2 + \frac{a^2}{2} - \frac{4a^2}{3} \\ &= \frac{a^2}{6} \end{aligned}$$

Area bounded by S_2 and coordinate axis in first quadrant

$$= \frac{\pi a^2}{4}$$

Area bounded by S_3 and coordinate axis in first quadrant

$$= \frac{a^2}{2}$$

$$\therefore \text{Area bounded by } S_1 \text{ and } S_2, \alpha = 4 \left(\frac{\pi a^2}{4} - \frac{a^2}{6} \right) = a^2 \left(\pi - \frac{2}{3} \right)$$

$$\text{Area bounded by } S_1 \text{ and } S_3, \beta = 4 \left(\frac{a^2}{2} - \frac{a^2}{6} \right) = \frac{4a^2}{3}$$

$$\text{Area bounded by } S_2 \text{ and } S_3, \gamma = 4 \left(\frac{\pi a^2}{4} - \frac{a^2}{2} \right) = a^2 (\pi - 2)$$

$$\text{Also, } \frac{\beta}{\gamma} = \frac{4}{3(\pi - 2)}$$

10. (1), (3)

x_1 and x_2 are the roots of the equation

$$x^2 + 2x - 3 = kx + 1$$

$$\text{or } x^2 + (2 - k)x - 4 = 0$$

$$\left. \begin{aligned} x_1 + x_2 &= k - 2 \\ x_1 x_2 &= -4 \end{aligned} \right\}$$

$$\begin{aligned} A &= \int_{x_1}^{x_2} [(kx + 1) - (x^2 + 2x - 3)] dx \\ &= \left[(k-2) \frac{x^2}{2} - \frac{x^3}{3} + 4x \right]_{x_1}^{x_2} \\ &= \left[(k-2) \frac{x_2^2 - x_1^2}{2} - \frac{1}{3} (x_2^3 - x_1^3) + 4(x_2 - x_1) \right] \\ &= (x_2 - x_1) \left[\frac{(k-2)^2}{2} - \frac{1}{3} ((x_2 - x_1)^2 - x_1 x_2) + 4 \right] \\ &= \sqrt{(x_2 + x_1)^2 - 4x_1 x_2} \left[\frac{(k-2)^2}{2} - \frac{1}{3} ((k-2)^2 + 4) + 4 \right] \\ &= \frac{\sqrt{(k-2)^2 + 16}}{6} \left[\frac{1}{6} (k-2)^2 + \frac{8}{3} \right] \\ &= \frac{[(k-2)^2 + 16]^{3/2}}{6} \end{aligned}$$

which is least when $k = 2$ and $A_{\text{least}} = 32/3$ sq. units

Linked Comprehension Type

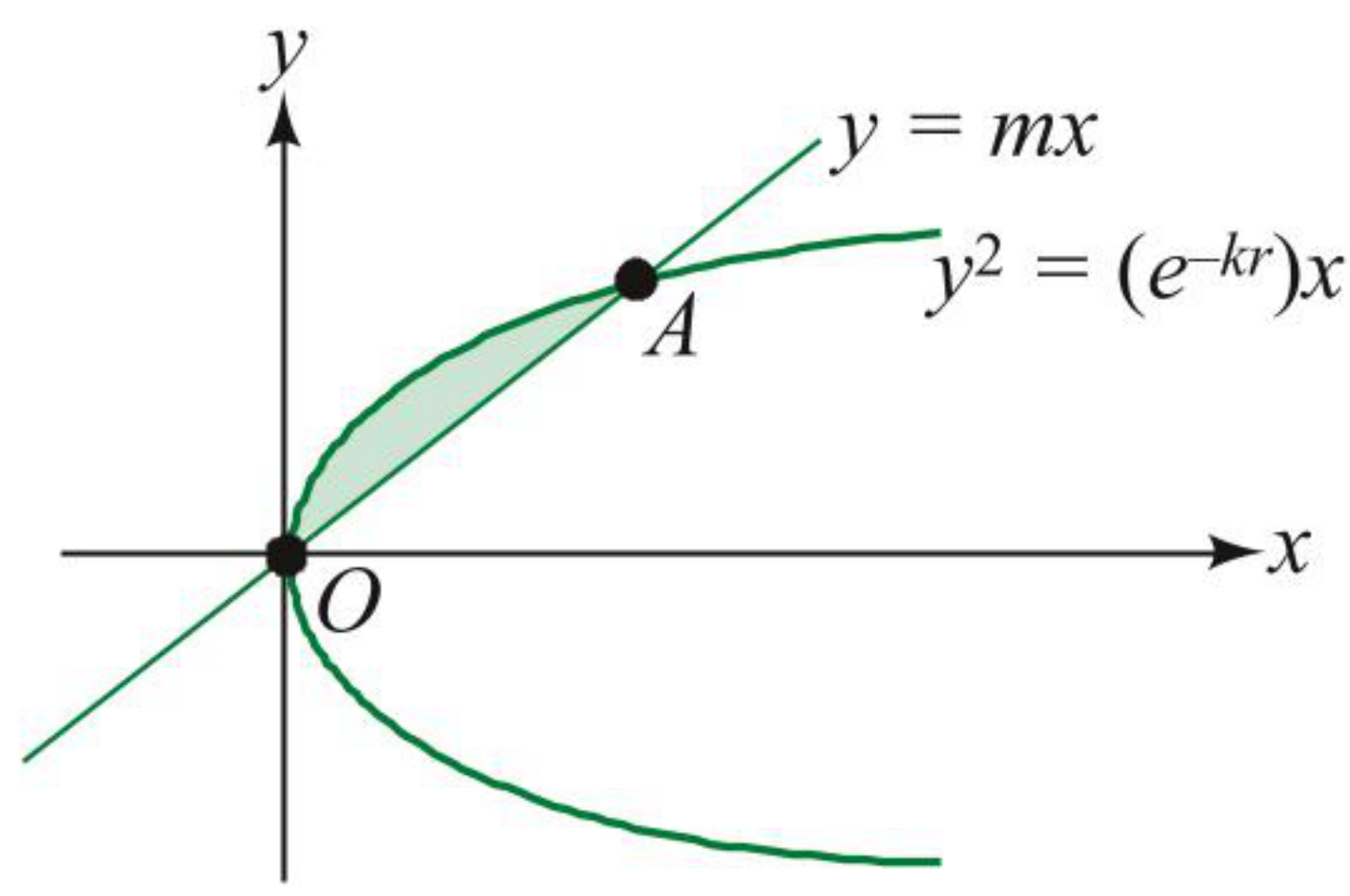
For Problems 1 and 2

1 (2), 2. (3)

Solving the two equations, we get

$$m^2 x^2 = (e^{-kr}) x$$

$$x_1 = 0, x_2 = \frac{e^{-kr}}{m^2}.$$



$$\begin{aligned}\text{So, } A_r &= \int_0^{x_2} \left(e^{-\frac{kr}{2}} \sqrt{x} - mx \right) dx \\ &= \frac{2}{3} e^{-kr/2} x_2^{3/2} - m \frac{x_2^2}{2} \\ &= \frac{2}{3} e^{-kr/2} \frac{e^{-3kr/2}}{m^3} - \frac{m}{2} \frac{e^{-2kr}}{m^4} = \frac{e^{-2kr}}{6m^3}.\end{aligned}$$

$$\text{Now, } \frac{A_{r+1}}{A_r} = \frac{e^{-2k(r+1)}}{e^{-2kr}} = e^{-2k} = \text{constant}.$$

So, the sequence $A_1, A_2, A_3, \dots$ is in G.P.

$$\text{Sum of } n \text{ terms} = \frac{e^{-2k}}{6m^3} \frac{e^{-2nk} - 1}{e^{-2k} - 1} = \frac{1}{6m^3} \frac{e^{-2nk} - 1}{1 - e^{2k}}$$

$$\begin{aligned}\text{Sum to infinite terms} &= A_1 \frac{1}{1 - e^{-2k}} \\ &= \frac{e^{-2k}}{6m^3} \times \frac{e^{2k}}{e^{2k} - 1} = \frac{1}{6m^3(e^{2k} - 1)}.\end{aligned}$$

For Problems 3–5

$$3. (4) f(x) = \frac{x^3}{3} - x^2 + a$$

$$f'(x) = x^2 - 2x = x(x-2) < 0 \quad (\text{note that } f(x) \text{ is monotonic in } (0, 2))$$

Hence, for the minimum and $f(x)$ must cross the x -axis at $\frac{0+2}{2} = 1$.

$$\text{Hence, } f(1) = \frac{1}{3} - 1 + a = 0$$

$$\Rightarrow a = \frac{2}{3}.$$

$$4. (3) f(x) = x^3 + 3x^2 + x + a$$

$$f'(x) = 3x^2 + 6x + 1 = 0$$

$$\Rightarrow x = -1 \pm \frac{\sqrt{6}}{3}.$$

Hence, $f(x)$ cuts the x -axis at

$$\frac{1}{2} \left[\left(-1 + \frac{\sqrt{6}}{3} \right) + \left(-1 - \frac{\sqrt{6}}{3} \right) \right] = -1.$$

$$f(-1) = -1 + 3 - 1 + a = 0$$

$$a = -1.$$

$$5. (1) f(x) = \sin x + \cos x$$

$$\text{or } \frac{df(x)}{dx} = \cos x - \sin x$$

$$\text{If } \frac{df(x)}{dx} = 0, \text{ then } \cos x = \sin x \Rightarrow x = \frac{\pi}{4} \text{ and } x = \frac{5\pi}{4}$$

(considering any two of consecutive points of extremum).

For minimum area bounded by $y = f(x)$ and $y = a$, between

$$x = \frac{\pi}{4} \text{ and } x = \frac{5\pi}{4}, \text{ graphs of } g(x) \text{ must cut } y = a \text{ at}$$

$$c = \frac{\frac{\pi}{4} + \frac{5\pi}{4}}{2} = \frac{3\pi}{4}.$$

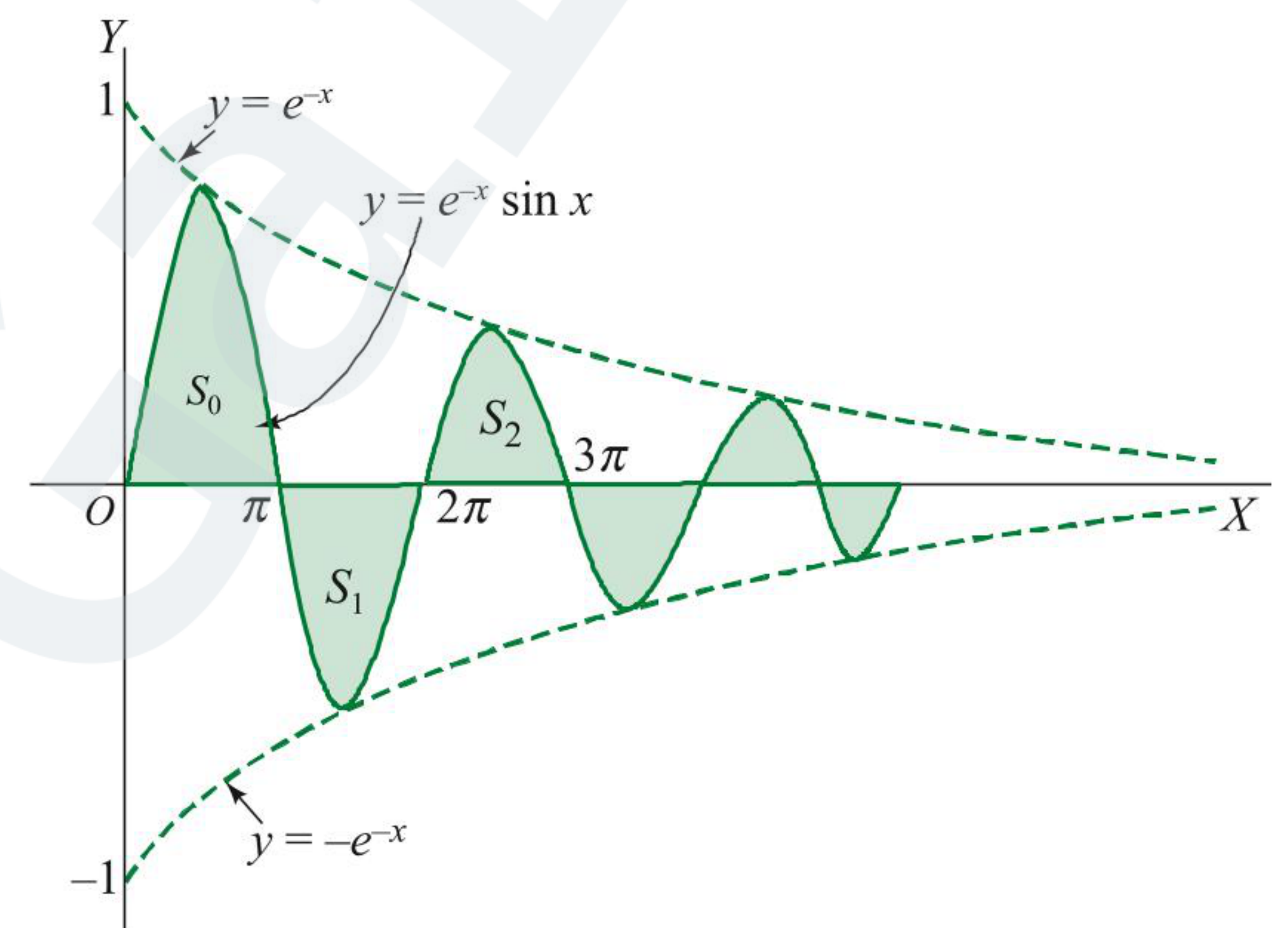
$$a = f\left(\frac{3\pi}{4}\right)$$

$$\Rightarrow a = \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = 0.$$

For Problems 6–8

6. (1), 7. (3), 8. (2)

Since $-1 \leq \sin x \leq 1$, the curve $y = e^{-x} \sin x$ is bounded by the curves $y = e^{-x}$ and $y = -e^{-x}$.



Also, the curve $y = e^{-x} \sin x$ intersects the positive semi-axis OX at the points where $\sin x = 0$, where $x_n = n\pi$, $n \in \mathbb{Z}$.

Also $|y_n| = |y \text{ coordinate in the half-wave } S_n|$

$$= (-1)^n e^{-x} \sin x, \text{ and in } S_n, n\pi \leq x \leq (n+1)\pi$$

$$\therefore S_n = (-1)^n \int_{n\pi}^{(n+1)\pi} e^{-x} \sin x dx$$

$$= \frac{(-1)^{n+1}}{2} \left[e^{-x} (-\sin x + \cos x) \right]_{n\pi}^{(n+1)\pi}$$

$$= \frac{(-1)^{n+1}}{2} \left[e^{-(n+1)\pi} (-1)^{n+1} - e^{-n\pi} (-1)^n \right]$$

$$= \frac{e^{-n\pi}}{2} (1 + e^\pi)$$

$$\Rightarrow \frac{S_{n+1}}{S_n} = e^{-\pi}$$

$$\text{and } S_0 = \frac{1}{2} (1 + e^\pi).$$

Therefore, the sequence $S_0, S_1, S_2, \dots$ forms an infinite G.P. with common ratio $e^{-\pi}$. we have

$$\sum_{n=0}^{\infty} S_n = \frac{\frac{1}{2}(1 + e^\pi)}{1 - e^{-\pi}}$$

For Problems 9–11

9. (2) Given

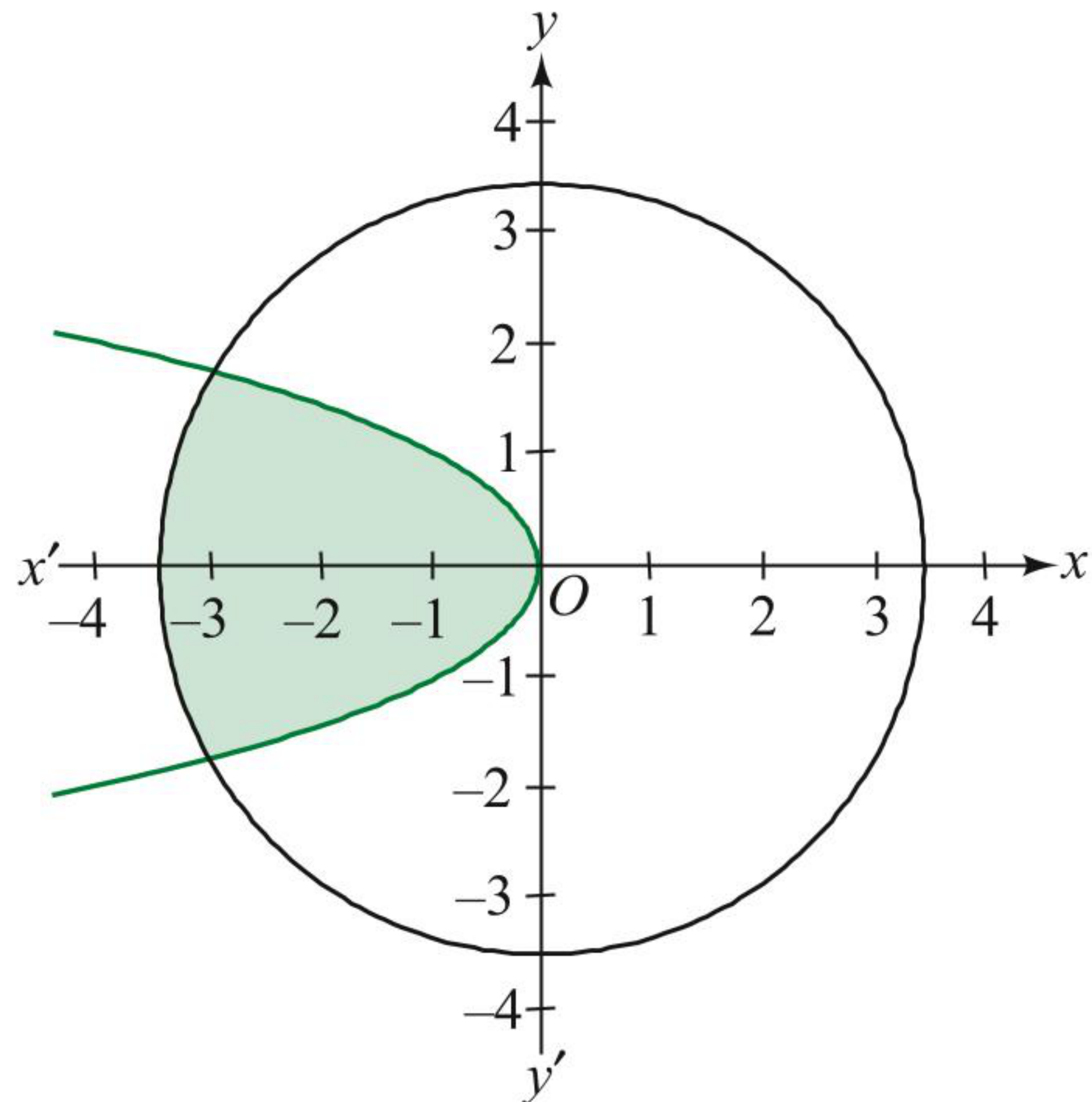
$$\begin{aligned}(x-y)f(x+y) - (x+y)f(x-y) &= 4xy(x^2 - y^2) \\ &= (x^2 - y^2) [(x+y)^2 - (x-y)^2] \\ &= (x-y)(x+y)^3 - (x+y)(x-y)^3\end{aligned}$$

$$\Rightarrow f(x+y) = (x+y)^3 \Rightarrow f(x) = x^3, f(y) = y^3$$

Now equations of given curves are

$$y^2 + x = 0 \quad \dots(1)$$

$$x^2 + y^2 = 12 \quad \dots(2)$$



Solving equations (1) and (2), we get $x = -3, y = \pm\sqrt{3}$

The area bounded by curves

$$A = 2 \left[\int_{-2\sqrt{3}}^{-3} \sqrt{12-x^2} dx + \left| \int_{-3}^0 \sqrt{-x} dx \right| \right]$$

$$I_1 = 2 \int_{-2\sqrt{3}}^{-3} \sqrt{12-x^2} dx = 2 \int_{-\pi/2}^{-\pi/3} 12 \cos^2 \theta d\theta$$

$$= 12 \left[\int_{-\pi/2}^{-\pi/3} (1 + \cos 2\theta) d\theta \right]$$

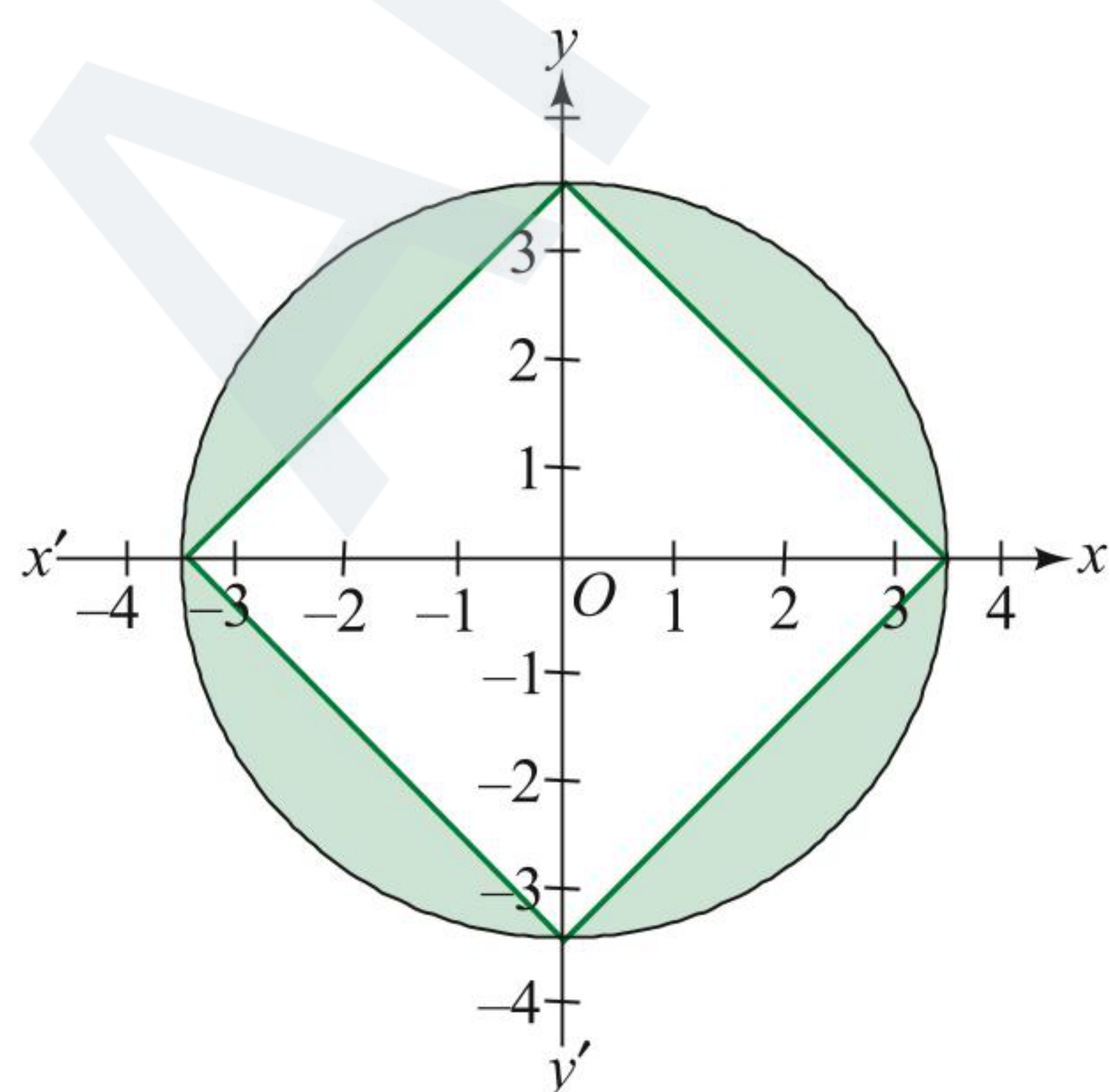
$$= 12 \left[\theta + \frac{\sin \theta}{2} \right]_{-\pi/2}^{-\pi/3} = 12 \left[-\frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{2} \right]$$

$$= 12 \left[\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right] = 2\pi - 3\sqrt{3}.$$

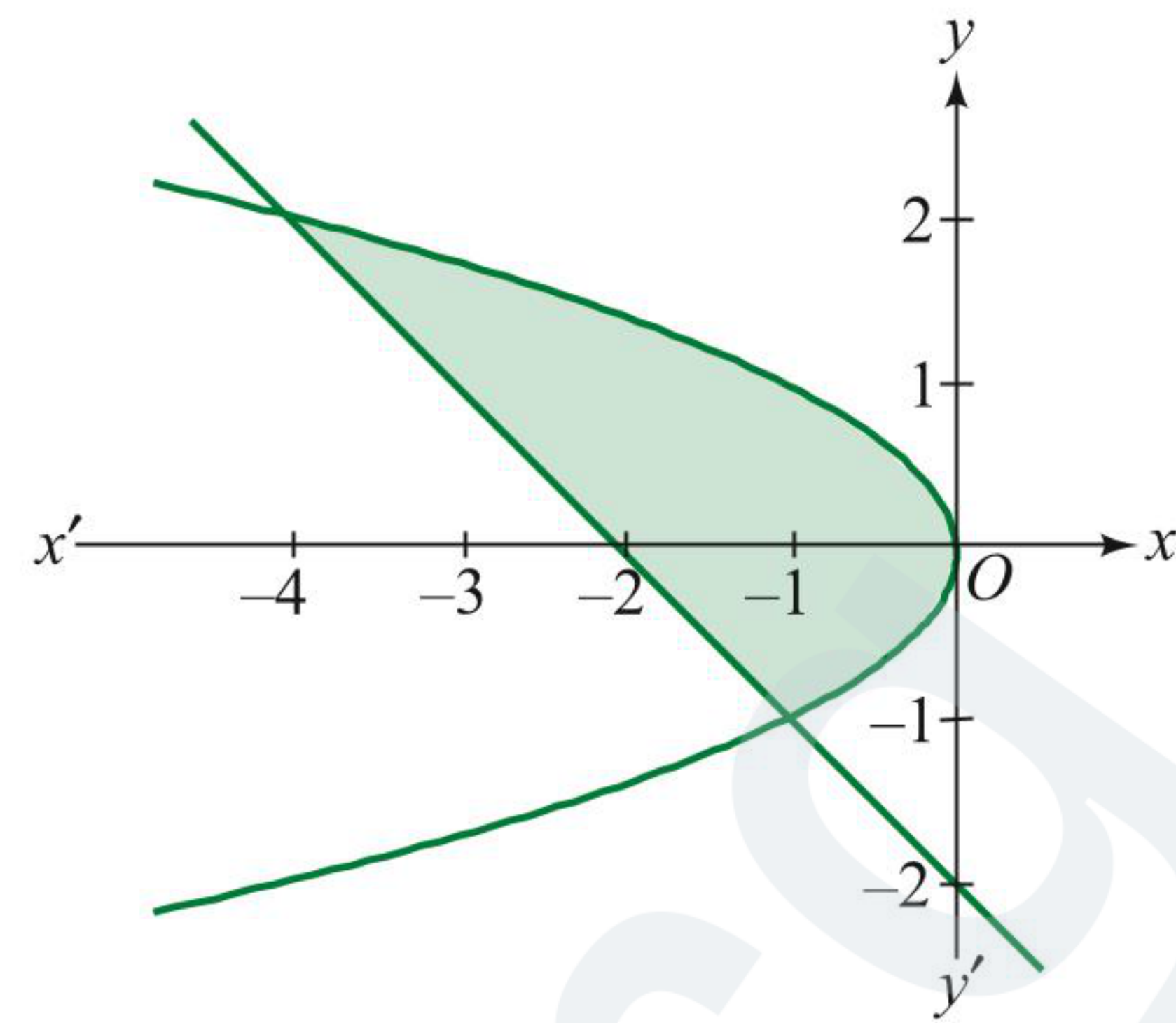
$$I_2 = 2 \int_{-3}^0 \sqrt{-x} dx = \frac{2[(-x)^{3/2}]_{-3}^0}{-3/2} = -\frac{4}{3}[0 - 3^{3/2}] = 4\sqrt{3}.$$

$$A = 2\pi - 3\sqrt{3} + 4\sqrt{3} = 2\pi + \sqrt{3} \text{ sq. units.}$$

10. (1) Required area = Area of circle – Area of square
 $= 12\pi - 24 \text{ sq. units.}$



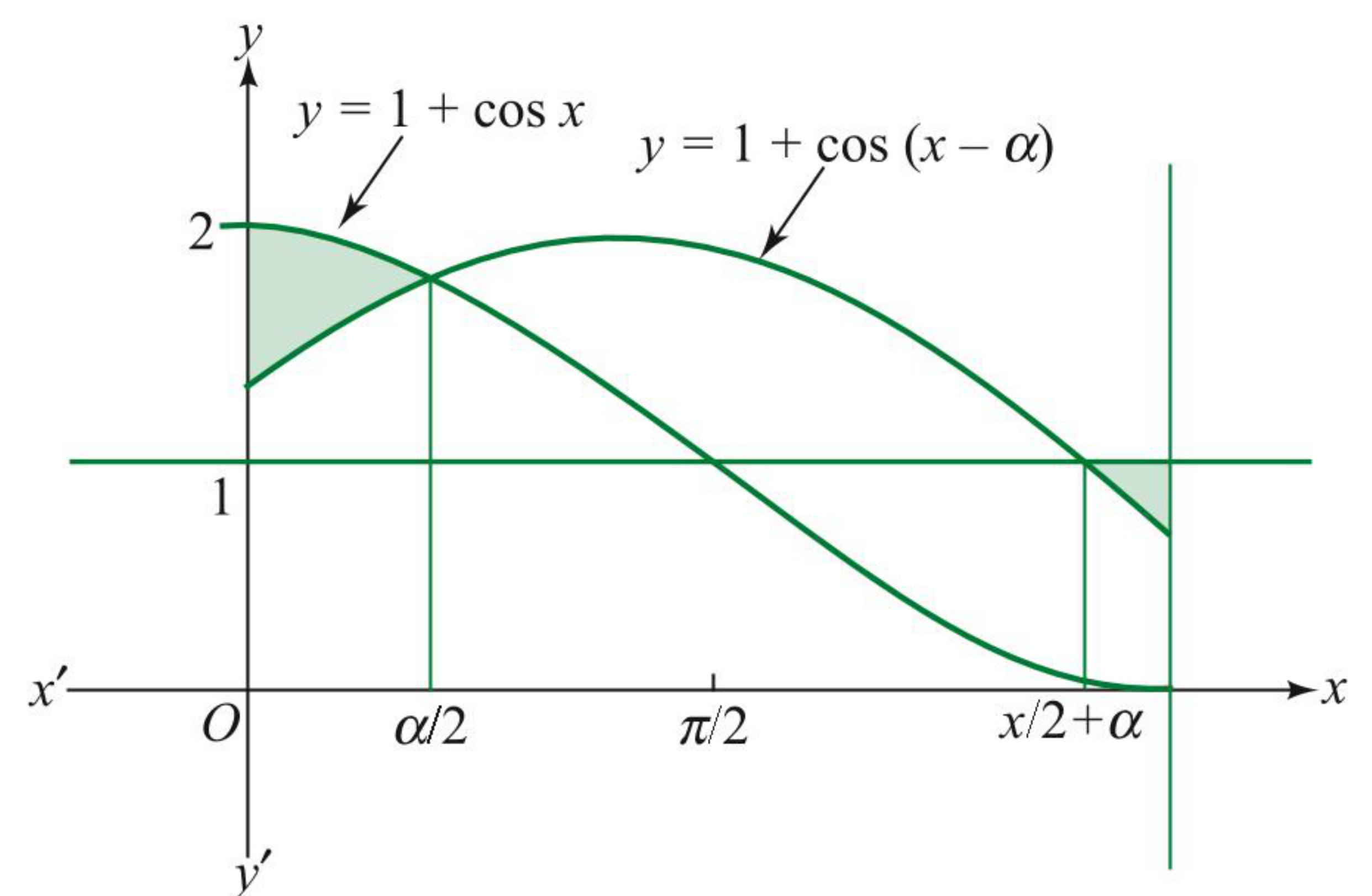
11. (3)



$$\begin{aligned} \text{Required area} &= \int_{-1}^2 (-y^2 - (-y-2)) dy \\ &= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 \\ &= \left[\frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right] \\ &= 9/2 \text{ sq. units.} \end{aligned}$$

For Problems 12 and 13

12. (3)



$$1 + \cos x = 1 + \cos(x - \alpha)$$

$$\text{or } x = \alpha - x \text{ or } x = \frac{\alpha}{2}$$

$$\text{Now } \int_0^{\alpha/2} ((1 + \cos x) - (1 + \cos(x - \alpha))) dx$$

$$= - \int_{\frac{\pi}{2} + \alpha}^{\pi} (1 - (1 + \cos(x - \alpha))) dx$$

$$\text{or } [\sin x - \sin(x - \alpha)]_0^{\alpha/2} = [\sin(x - \alpha)]_{\pi}^{\frac{\pi}{2} + \alpha}$$

$$\text{or } \left[\sin \frac{\alpha}{2} - \sin \left(-\frac{\alpha}{2} \right) \right] - [0 - \sin(-\alpha)]$$

$$= \sin \left(\frac{\pi}{2} \right) - \sin(\pi - \alpha)$$

$$\text{or } 2 \sin \frac{\alpha}{2} - \sin \alpha = 1 - \sin \alpha$$

$$\text{Hence, } 2 \sin \frac{\alpha}{2} = 1 \text{ or } \alpha = \frac{\pi}{3}.$$

$$\begin{aligned}
 13. (2) \quad & \int_0^{\pi/6} \left((1 + \cos x) - \left(1 + \cos \left(x - \frac{\pi}{3} \right) \right) \right) dx \\
 & + \int_{\pi/6}^{\pi} \left(\left(1 + \cos \left(x - \frac{\pi}{3} \right) \right) - (1 + \cos x) \right) dx \\
 & = \left[\sin x - \sin \left(x - \frac{\pi}{3} \right) \right]_0^{\pi/6} + \left[\sin \left(x - \frac{\pi}{3} \right) - \sin x \right]_{\pi/6}^{\pi} \\
 & = \left[\left(\frac{1}{2} + \frac{1}{2} \right) - \frac{\sqrt{3}}{2} \right] + \left[\frac{\sqrt{3}}{2} - \left(-\frac{1}{2} - \frac{1}{2} \right) \right] \\
 & = 2 \text{ sq. units.}
 \end{aligned}$$

For Problems 14–1614. (1) For $-1 \leq x < 0$

$$\begin{aligned}
 (y - e^{\sin^{-1} x})^2 &= 2 - x^2 \\
 y &= e^{\sin^{-1} x} \pm \sqrt{2 - x^2} \\
 A &= \int_{-1}^0 \left(e^{\sin^{-1} x} + \sqrt{2 - x^2} \right) - \left(e^{\sin^{-1} x} - \sqrt{2 - x^2} \right) dx \\
 &= 2 \int_{-1}^0 \sqrt{2 - x^2} dx \\
 &= 2 \left(\frac{1}{2} x \sqrt{2 - x^2} \Big|_{-1}^0 + \frac{2}{2} \sin^{-1} \frac{x}{\sqrt{2}} \Big|_{-1}^0 \right) \\
 &= \left[1 + 2 \left(0 - \left(-\frac{\pi}{4} \right) \right) \right] \\
 &= \frac{\pi}{2} + 1 \text{ sq. units.}
 \end{aligned}$$

For $0 \leq x < 1$, $y = \sin^{-1} x \pm \sqrt{1 - x^2}$

$$\begin{aligned}
 A &= 2 \int_0^1 \sqrt{1 - x^2} dx \\
 &= 2 \left[\frac{x}{2} \sqrt{1 - x^2} \Big|_0^1 + \frac{1}{2} \sin^{-1} \frac{x}{1} \Big|_0^1 \right] \\
 &= 0 + \sin^{-1}(1) = \frac{\pi}{2} \text{ sq. units.}
 \end{aligned}$$

$$\text{Total area} = \left(\frac{\pi}{2} + 1 \right) + \frac{\pi}{2} = \pi + 1.$$

$$15. (4) \quad \text{Ratio} = \frac{\frac{\pi}{2} + 1}{\frac{\pi}{2}} = \frac{\pi + 2}{\pi}.$$

$$\begin{aligned}
 16. (1) \quad A &= 2 \int_0^{1/2} \sqrt{1 - x^2} dx \\
 &= 2 \left[\frac{x}{2} \sqrt{1 - x^2} \Big|_0^{1/2} + \frac{1}{2} \sin^{-1} x \Big|_0^{1/2} \right] \\
 &= \frac{\sqrt{3}}{4} + \frac{\pi}{6} \text{ sq. unit.}
 \end{aligned}$$

For Problems 17–19

$$17. (2) \quad S = \left| -\int_0^{2\pi} a(1 - \cos t)a(1 - \cos t)dt \right|$$

$$\begin{aligned}
 &= \left| -a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \right| \\
 &= \left| -a^2 \int_0^{2\pi} \left(1 - 2\cos t + \left(\frac{1 + \cos 2t}{2} \right) \right) dt \right| \\
 &= \left| -\frac{a^2}{2} \int_0^{2\pi} (3 - 4\cos t + \cos 2t)dt \right| \\
 &= \left| -\frac{a^2}{2} [3t - 4\cos t + \cos 2t]_0^{2\pi} \right| \\
 &= |-3\pi a^2| = 3\pi a^2 \text{ sq. units.}
 \end{aligned}$$

$$\begin{aligned}
 18. (1) \quad & \int_0^6 \left(\frac{3}{2}t^2 - \frac{1}{2}t^3 + \frac{1}{24}t^4 \right) dt = \frac{3}{2} \times \frac{6^3}{3} \times \frac{1}{2} \times \frac{6^4}{4} \times \frac{1}{24} \times \frac{6^5}{5} \\
 &= \frac{6^3}{2} - \frac{6^4}{8} + \frac{6^4}{20} \\
 &= 6^4 \left(\frac{1}{12} - \frac{1}{8} + \frac{1}{20} \right) \\
 &= \frac{54}{5}. \\
 \therefore \quad & \frac{1}{2} \int_0^6 (xy' - yx')dx = \frac{1}{2} \times \frac{54}{5} = \frac{27}{5} \text{ sq. units.}
 \end{aligned}$$

$$19. (3) \quad \frac{dx}{dt} = 1 - 3t^2 \text{ and } \frac{dy}{dt} = 1 - 4t^3.$$

$$\begin{aligned}
 \text{So, } x \frac{dy}{dt} - y \frac{dx}{dt} &= (t - t^3)(1 - 4t^3) - (1 - t^4)(1 - 3t^2) \\
 &= t^6 - 3t^4 - t^3 + 3t^2 + t - 1
 \end{aligned}$$

$$\begin{aligned}
 \therefore \quad \text{Required area} &= \frac{1}{2} \int_{-1}^1 (t^6 - 3t^4 - t^3 + 3t^2 + t - 1)dt \\
 &= \frac{16}{35} \text{ sq. units (taking absolute value).}
 \end{aligned}$$

For Problems 20 and 21

20. (3), 21. (2)

$$f(x) = \begin{cases} x^2 + ax + b; & x < -1 \\ 2x; & -1 \leq x \leq 1 \\ x^2 + ax + b; & x > 1 \end{cases}$$

 $\therefore f(x)$ is continuous at $x = -1$ and $x = 1$

$$\therefore (-1)^2 + a(-1) + b = -2 \Rightarrow b - a = -3 \quad \dots(1)$$

$$\text{and } 2 = (1)^2 + a(1) + b \Rightarrow a + b = 1 \quad \dots(2)$$

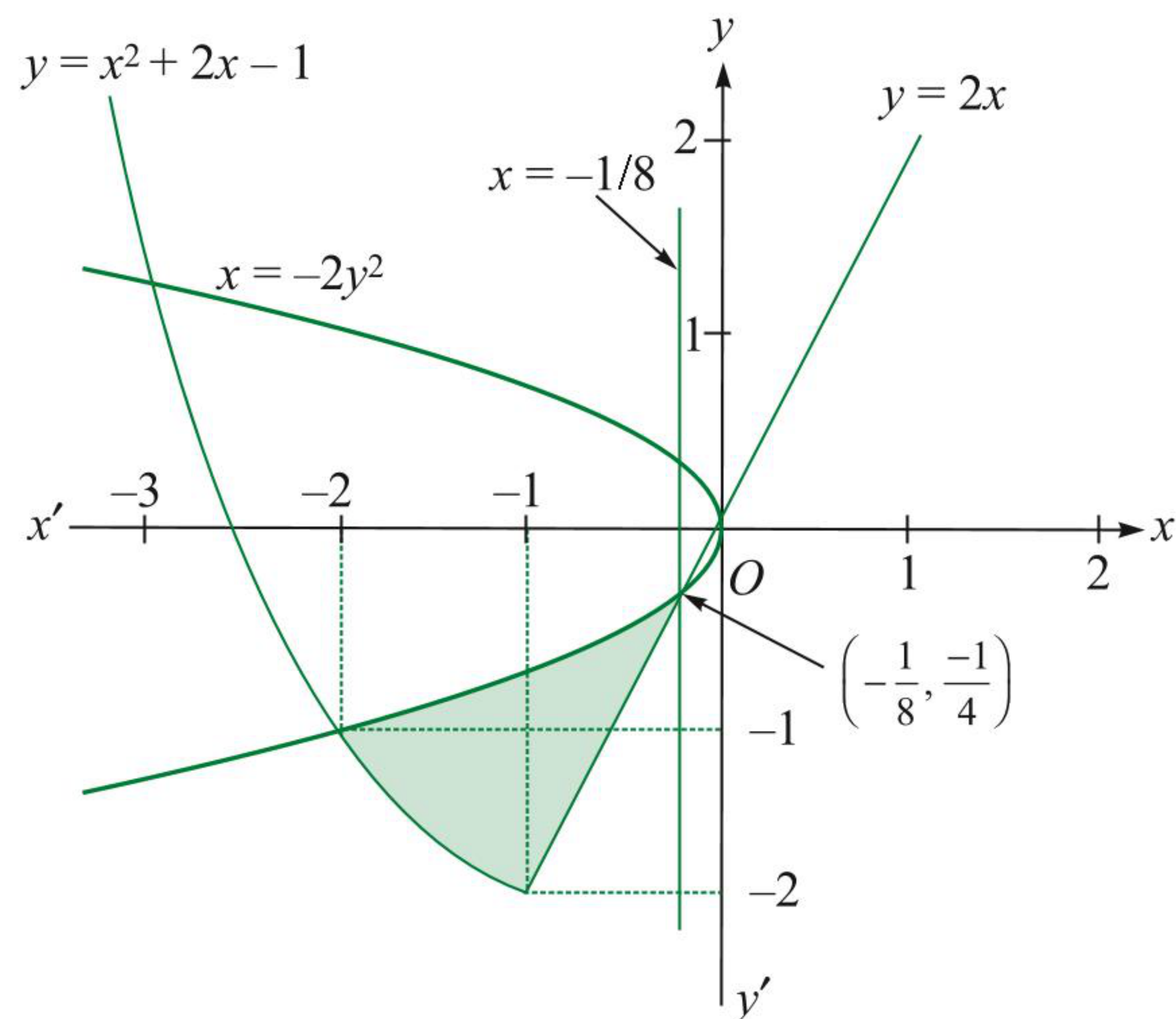
Solving (1) and (2), we get $a = 2$, $b = -1$

$$\therefore f(x) = \begin{cases} x^2 + 2x - 1; & x < -1 \\ 2x; & -1 \leq x \leq 1 \\ x^2 + 2x - 1; & x > 1 \end{cases}$$

Given curves are $y = f(x)$, $x = -2y^2$ and $8x + 1 = 0$ Solving $x = -2y^2$, $y = x^2 + 2x - 1$ ($x < -1$) we get $x = -2$ Also, $y = 2x$, $x = -2y^2$ meet at $(0, 0)$

$$y = 2x \text{ and } x = -1/8 \text{ meet at } \left(-\frac{1}{8}, \frac{-1}{4} \right)$$

The required area is the shaded region in the figure



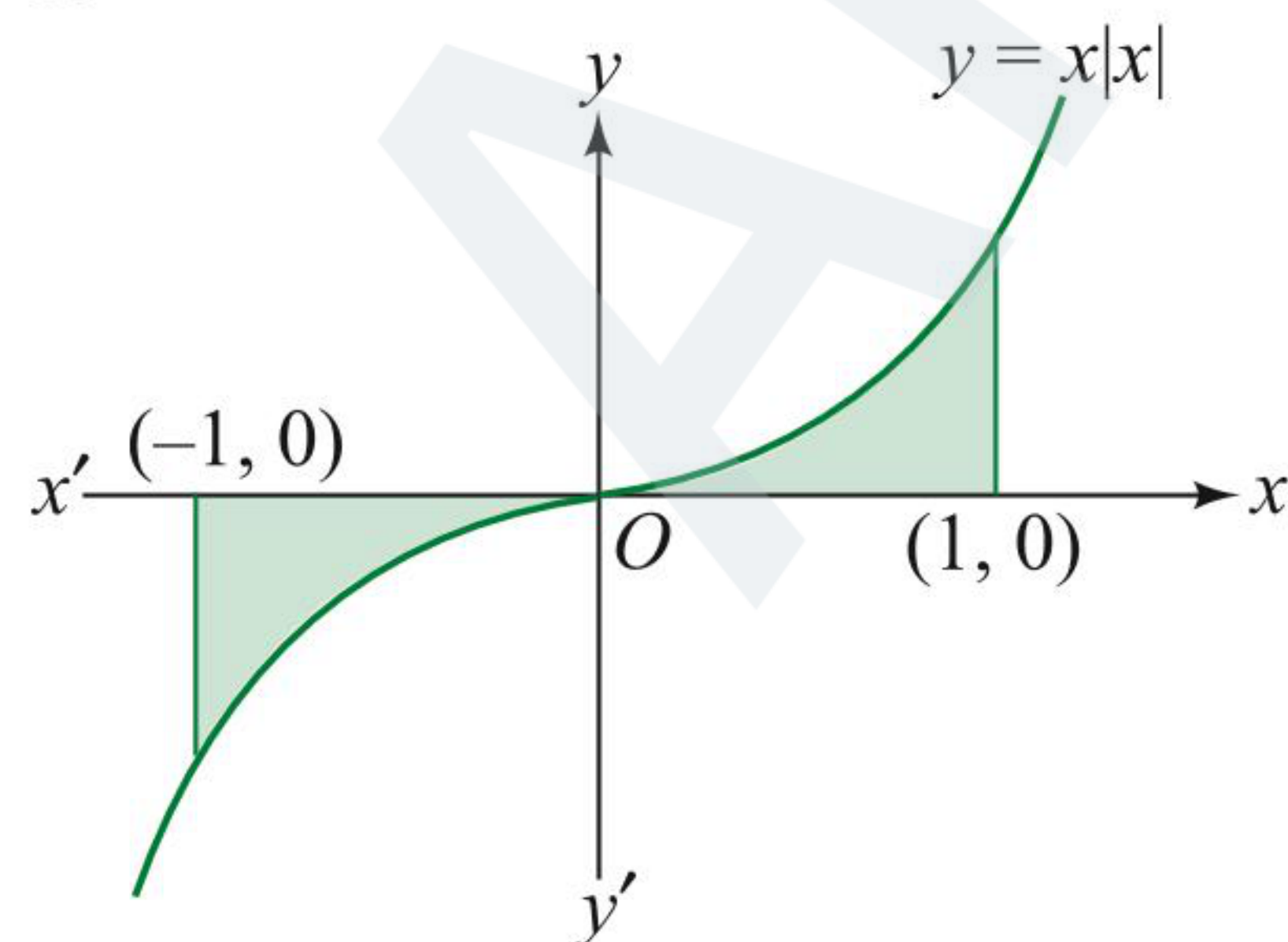
∴ Required area

$$\begin{aligned}
 &= \int_{-2}^{-1} \left[-\sqrt{\frac{-x}{2}} - (x^2 + 2x - 1) \right] dx \\
 &\quad + \int_{-1}^{-1/8} \left[-\sqrt{\frac{-x}{2}} - 2x \right] dx \\
 &= \left[\frac{1}{\sqrt{2}} \frac{2(-x)^{3/2}}{3} - \frac{x^3}{3} - x^2 + x \right]_{-2}^{-1} \\
 &\quad + \left[\frac{1}{\sqrt{2}} \frac{2(-x)^{3/2}}{3} - x^2 \right]_{-1}^{-1/8} \\
 &= \left(\frac{\sqrt{2}}{3} + \frac{1}{3} - 1 - 1 \right) - \left(\frac{4}{3} + \frac{8}{3} - 4 - 2 \right) \\
 &\quad + \left(\frac{\sqrt{2}}{3} \cdot \frac{1}{16\sqrt{2}} - \frac{1}{64} \right) - \left(\frac{\sqrt{2}}{3} - 1 \right) \\
 &= \left(\frac{\sqrt{2} - 5}{3} \right) - \left(\frac{4 + 8 - 18}{3} \right) + \left(\frac{4 - 3}{192} \right) - \left(\frac{\sqrt{2} - 3}{3} \right) \\
 &= \frac{257}{192} \text{ sq. units}
 \end{aligned}$$

Matrix Match Type

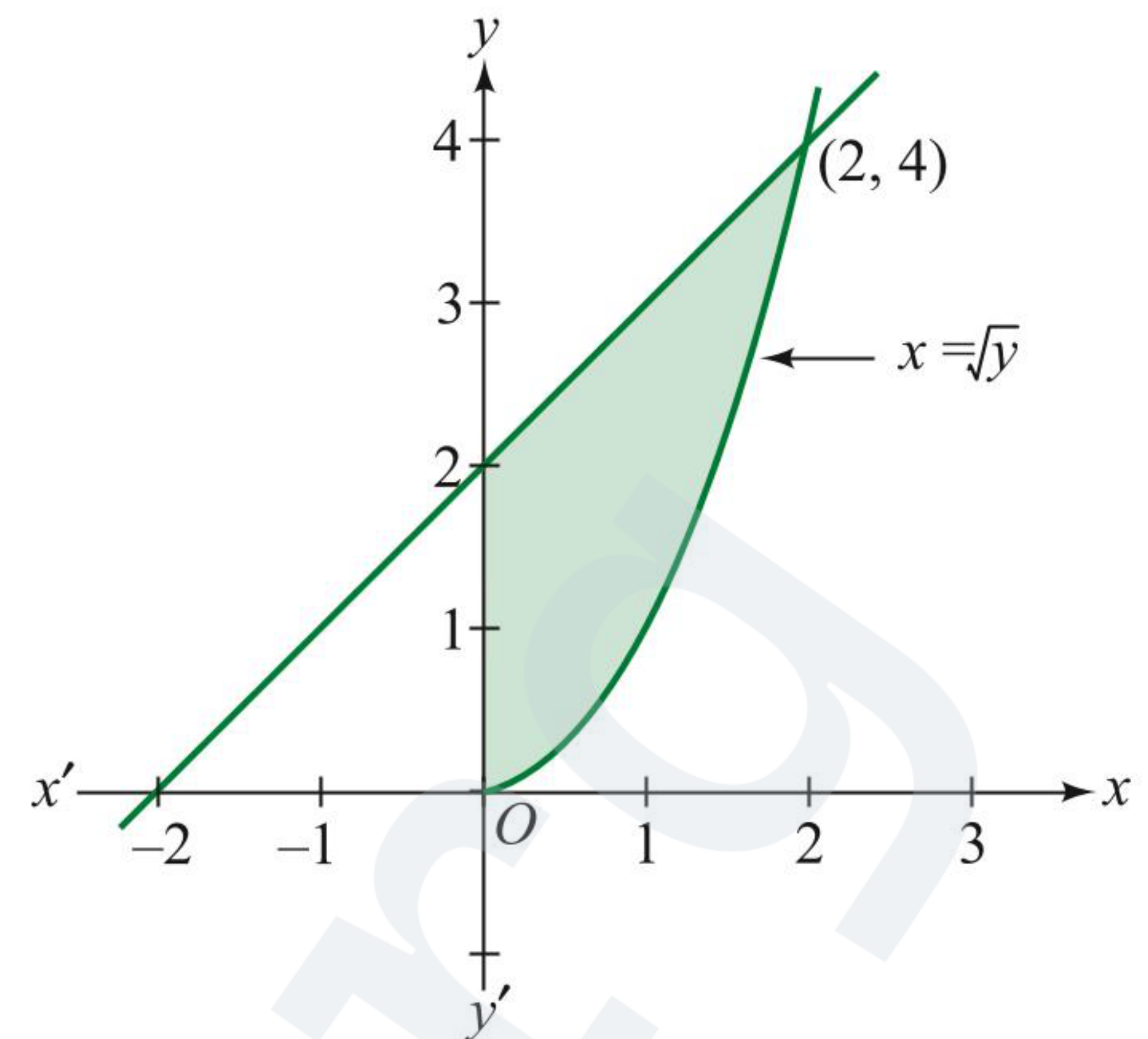
1. a → r; b → p; c → s; d → q

a.



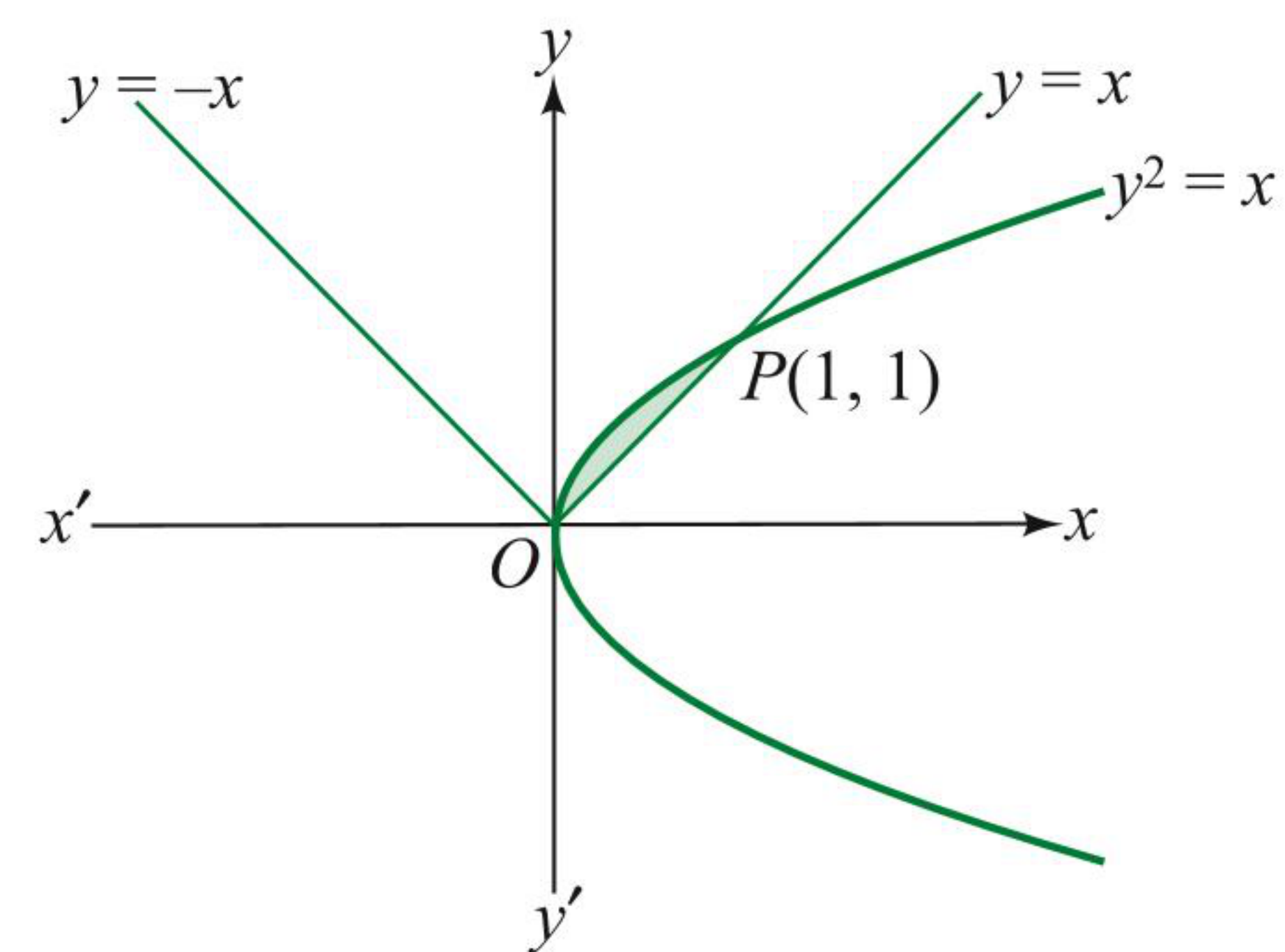
$$\begin{aligned}
 \text{Required area} &= 2 \int_0^1 x|x| dx \\
 &= 2 \left(\frac{x^3}{3} \right)_0^1 = \frac{2}{3}
 \end{aligned}$$

b.

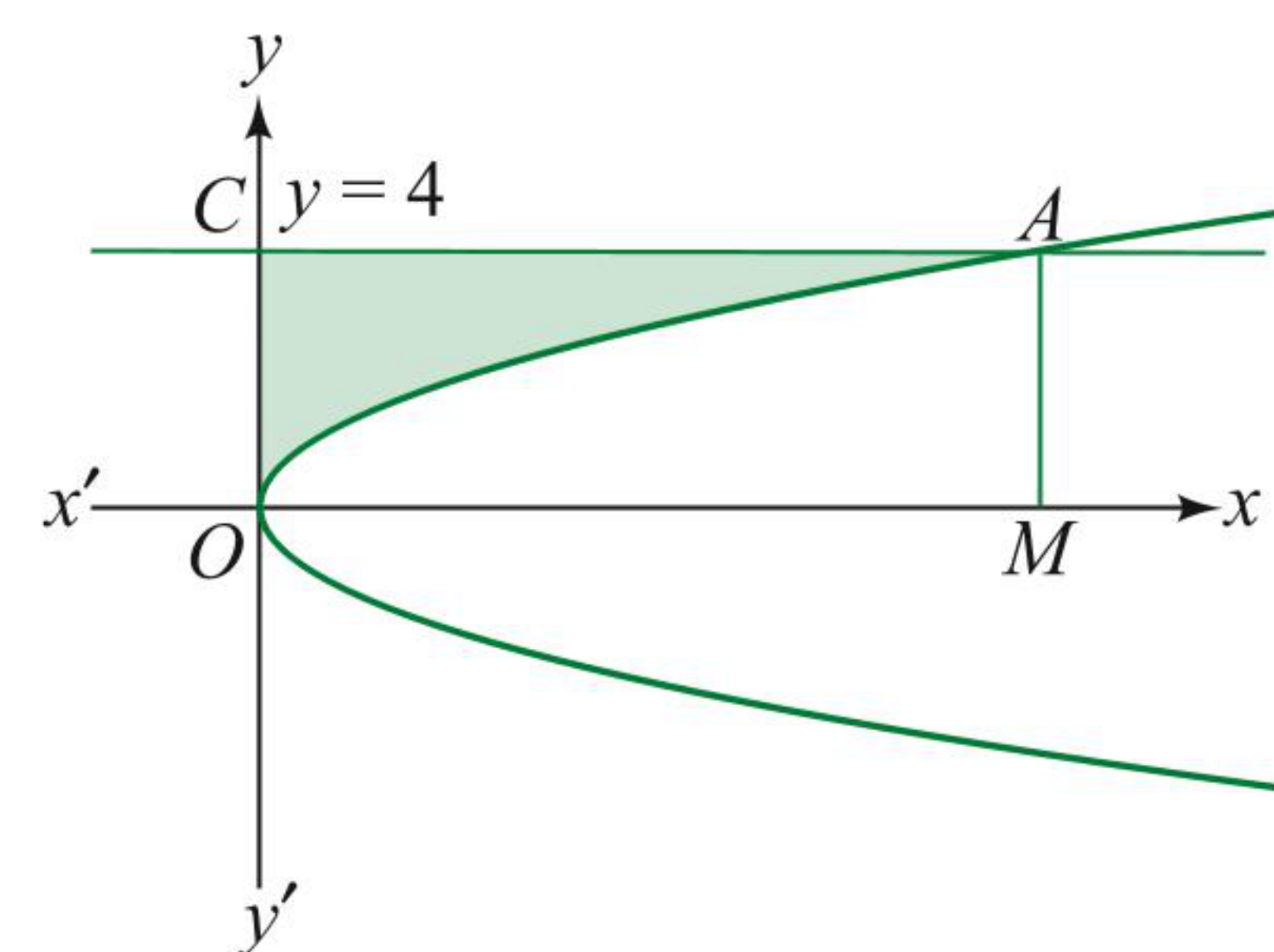


$$\begin{aligned}
 \text{Required area} &= \int_0^2 [(x + 2) - (x^2)] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_0^2 \\
 &= 2 + 4 - \frac{8}{3} = \frac{10}{3} \text{ sq. units.}
 \end{aligned}$$

$$\begin{aligned}
 \text{c. Reqd. area} &= \int_0^1 (\sqrt{x} - x) dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1 \\
 &= \left(\frac{1}{3/2} - \frac{1}{2} \right) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ sq. units.}
 \end{aligned}$$



d. $y = 4$ meets the parabola $y^2 = x$ at A is $(16, 4)$

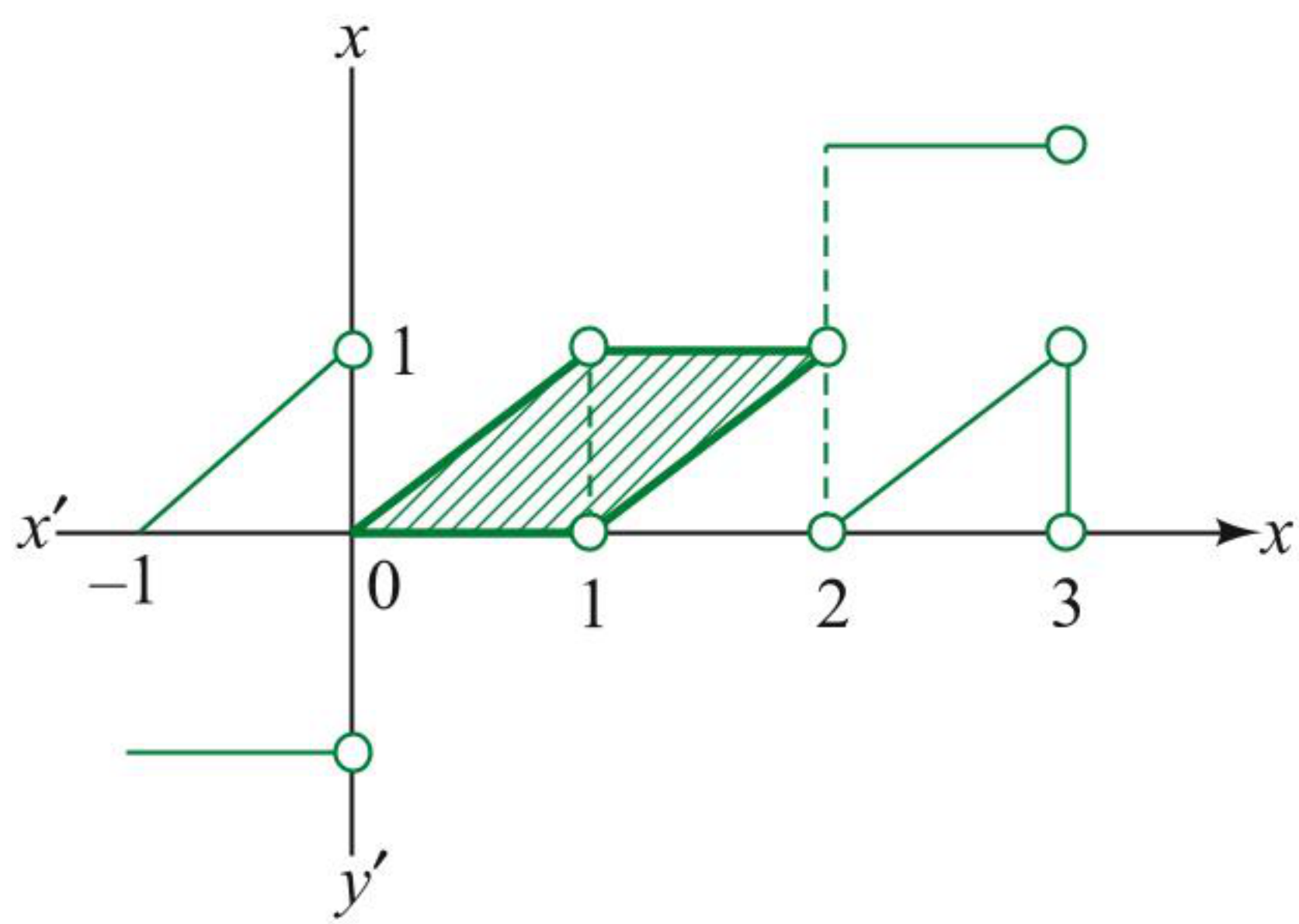


Required area = Area of rectangle OMAC - Area OMA

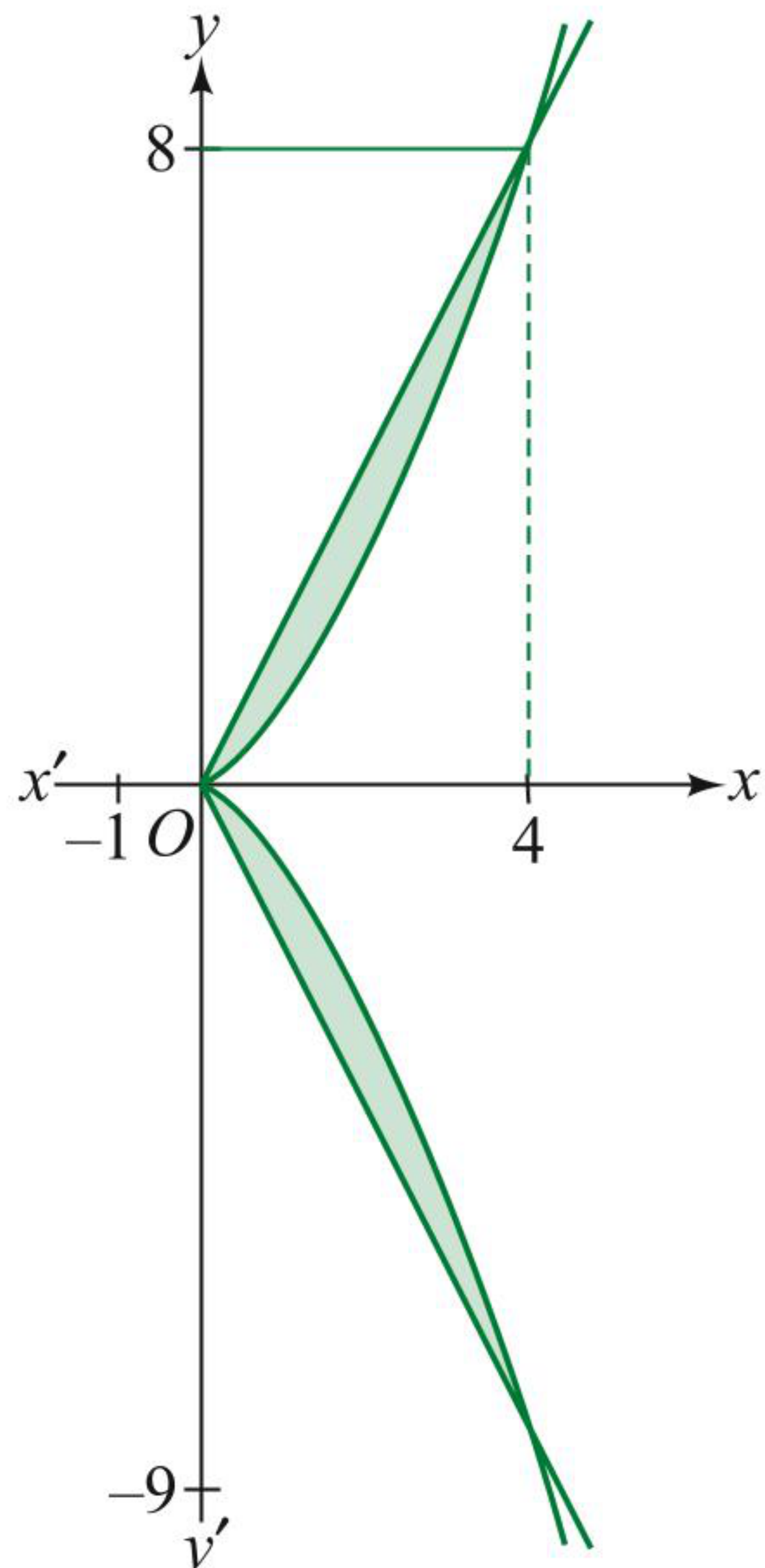
$$\begin{aligned}
 &= 4 \times 16 - \int_0^{16} \sqrt{x} dx = 64 - \left[\frac{x^{3/2}}{3/2} \right]_0^{16} \\
 &= 64 - \frac{2}{3} (4)^3 \\
 &= 64 - \frac{128}{3} = \frac{64}{3} \text{ sq. units.}
 \end{aligned}$$

2. a → q; b → p; c → s; d → r

$$\text{a. Area} = 2 \left(\frac{1}{2} \times 1 \times 1 \right) = 1 \text{ sq. units.}$$



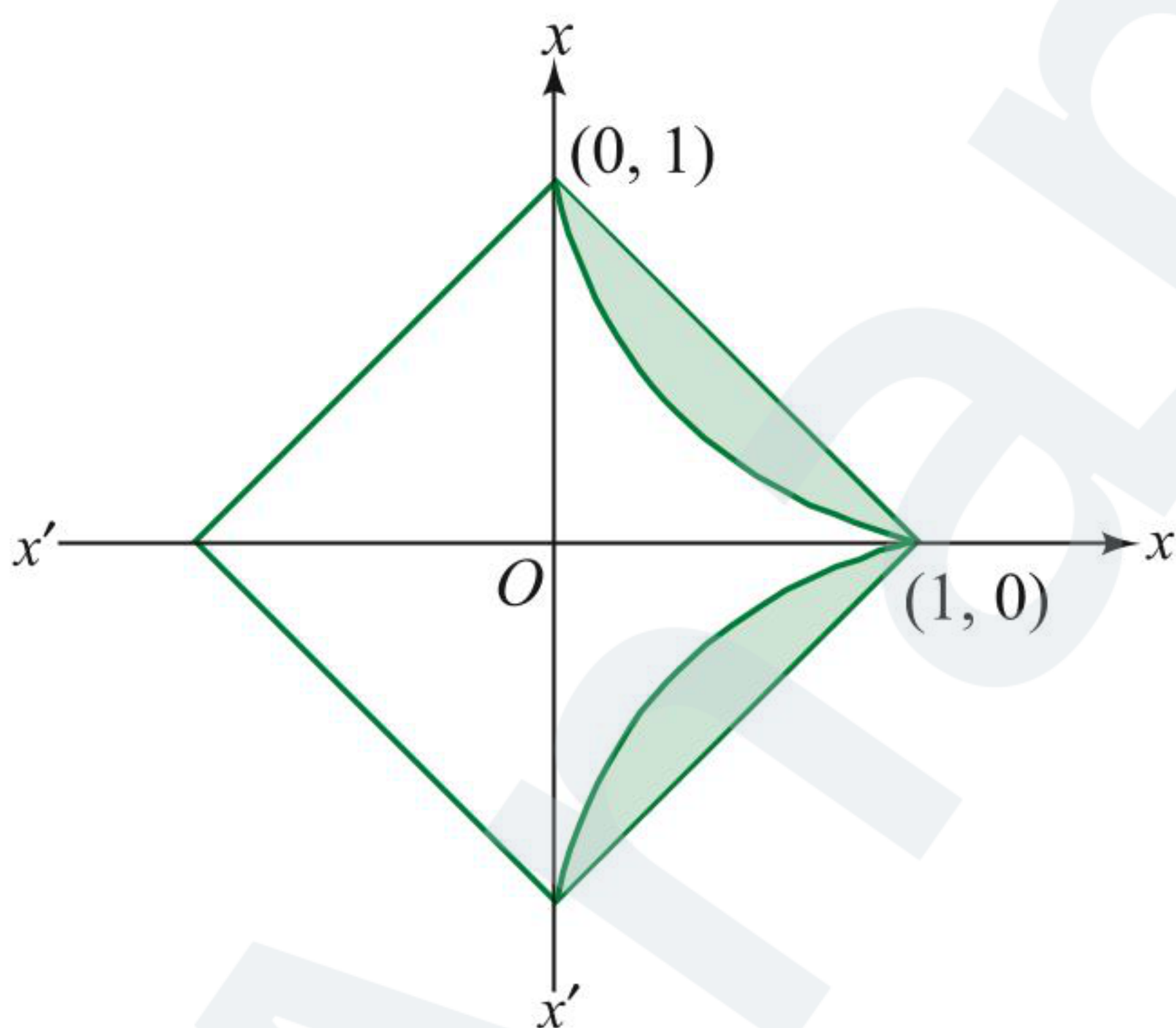
- b. $y^2 = x^3$ and $|y| = 2x$, both the curve are symmetric about y -axis



$$4x^2 = x^3 \text{ or } x = 0, 4.$$

$$\text{Required area} = 2 \int_0^4 (2x - x^{3/2}) dx = \frac{32}{5} \text{ sq. units.}$$

- c. $\sqrt{x} + \sqrt{|y|} = 1$



The curve is symmetrical about x -axis

$$\sqrt{|y|} = 1 - \sqrt{x} \text{ and } \sqrt{x} = 1 - \sqrt{|y|}$$

$$\Rightarrow \text{for } x > 0, y > 0 \quad \sqrt{y} = 1 - \sqrt{x}$$

$$\frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$\frac{dy}{dx} < 0, \text{ function is decreasing}$$

$$\text{Required area} = 2 \int_0^1 ((1-x) - (1-2\sqrt{x}+x)) dx$$

$$\begin{aligned} &= 4 \int_0^1 (\sqrt{x} - x) dx \\ &= 4 \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1 \\ &= 4 \left[\frac{2}{3} - \frac{1}{2} \right] \\ &= \frac{2}{3} \text{ sq. units.} \end{aligned}$$

- d. If $-8 < x < 8$, then $y = 2$.

If $x \in (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$, then $y = 3$, and so on

Intersection of $y = x - 1$ and $y = 2$. We get $x = 3 \in (-8, 8)$.

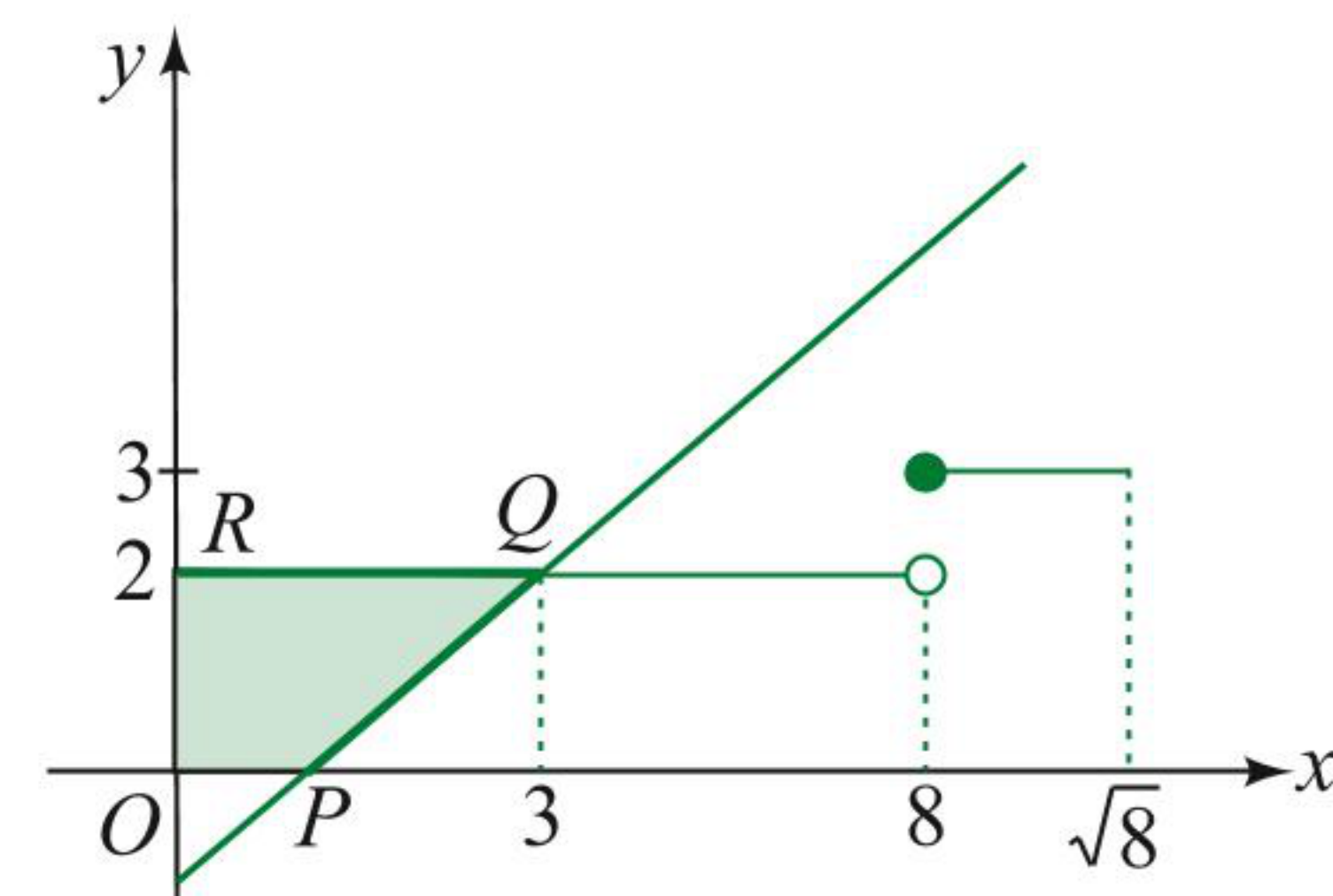
Intersection of $y = x - 1$ and $y = 3$.

We get $x = 4 \notin (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$.

Similarly, $y = x - 1$ will not intersect $y = \left[\frac{x^2}{64} + 2 \right]$ at any

other integral, except in the interval $x \in (-8, 8)$.

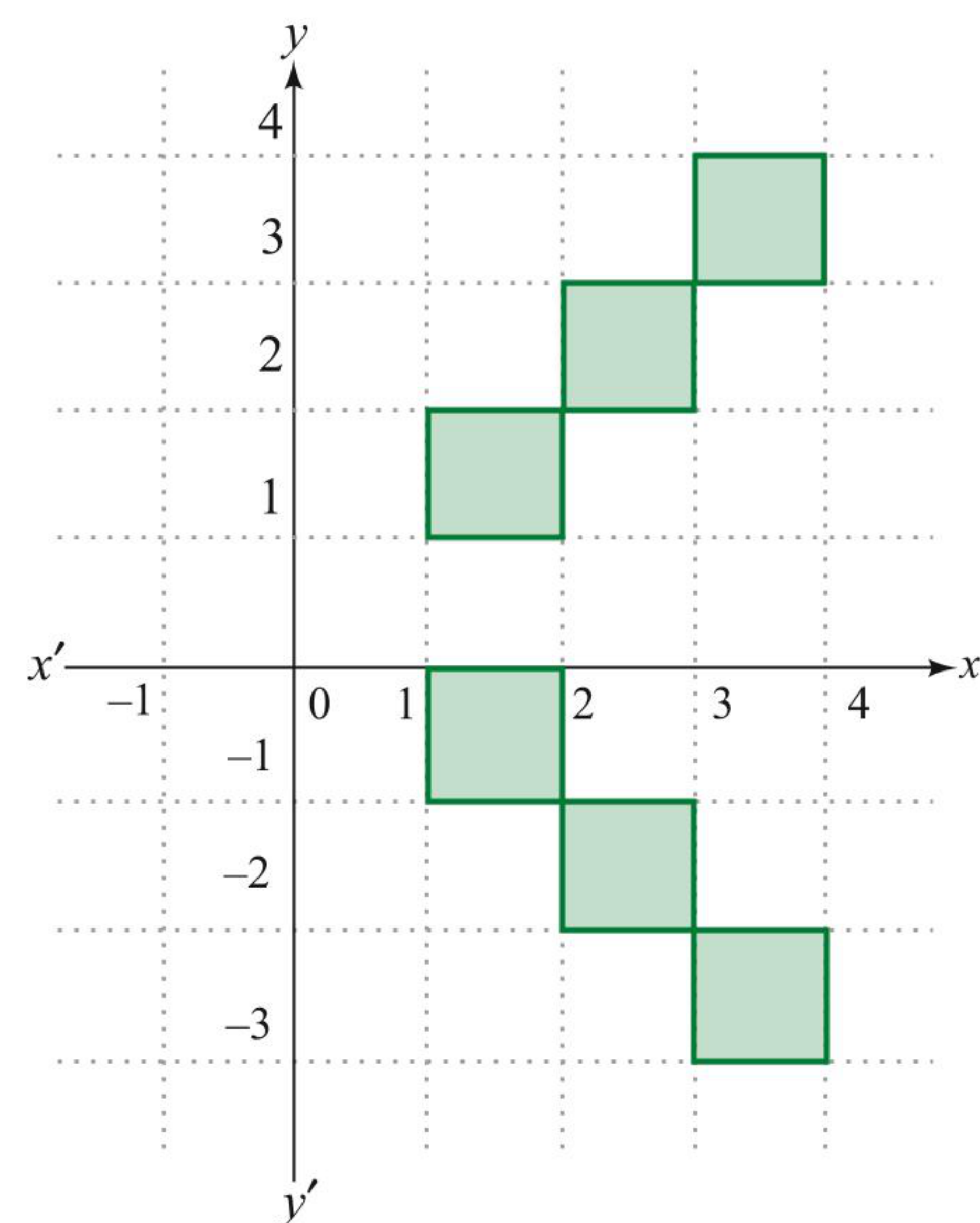
$$\begin{aligned} \text{Required area (shaded region)} &= 2 \times 3 - \frac{1}{2} \times 2 \times 2 \\ &= 4 \text{ sq. units.} \end{aligned}$$



3. a $\rightarrow$ q; b $\rightarrow$ s; c $\rightarrow$ p; d $\rightarrow$ p

- a. $[x]^2 = [y]^2$, where $1 \leq x \leq 4$

$$\Rightarrow [x] = \pm [y]$$



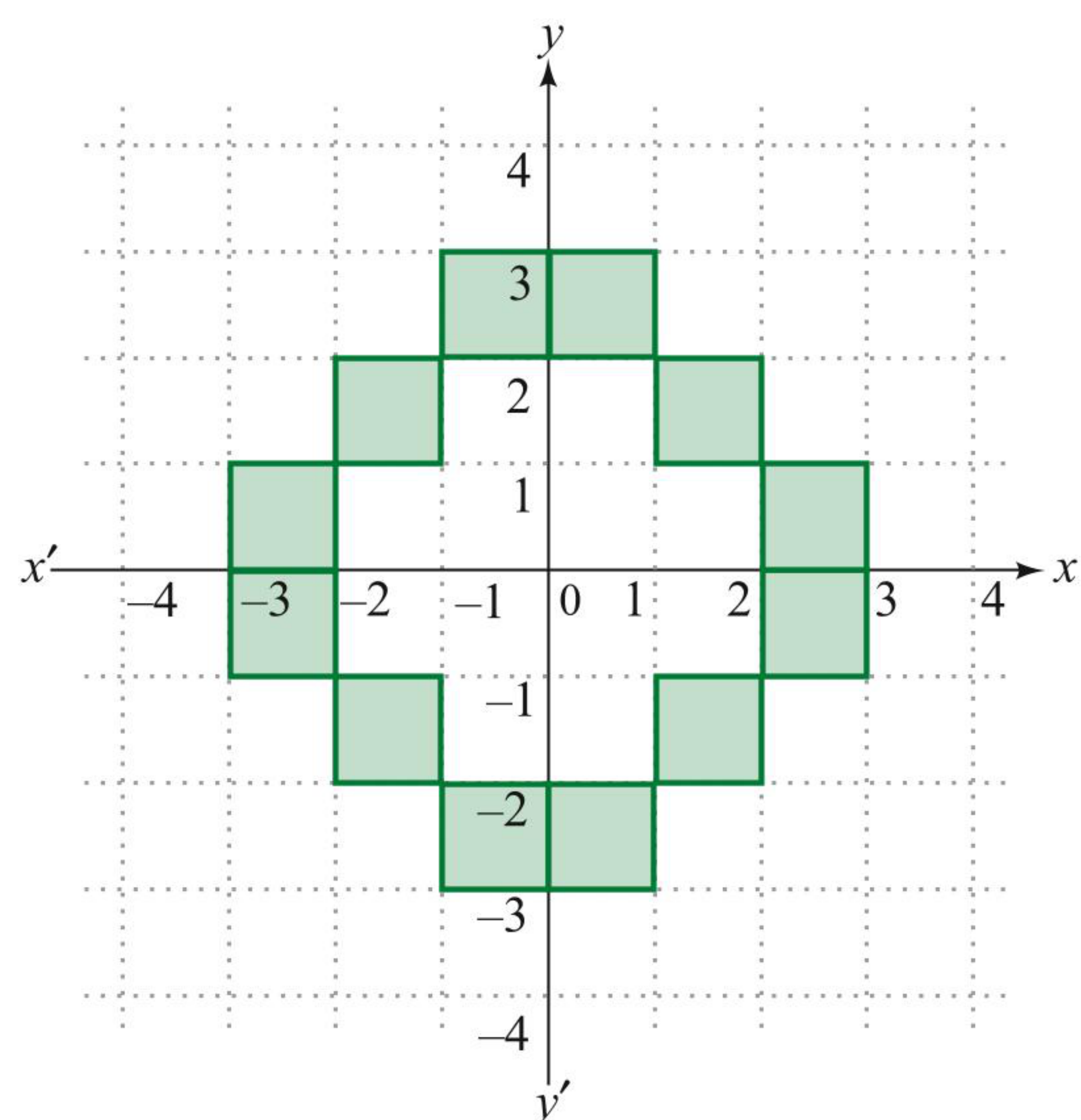
- b. $[|x|] + [|y|] = 2$

The graph is symmetrical about both x -axis and y -axis.

For $x, y > 0$; $[x] + [y] = 2$.

$$\Rightarrow [x] = 0 \text{ and } [y] = 2, [x] = 1 \text{ and}$$

$$[y] = 1 \text{ or } [x] = 2 \text{ and } [y] = 0.$$

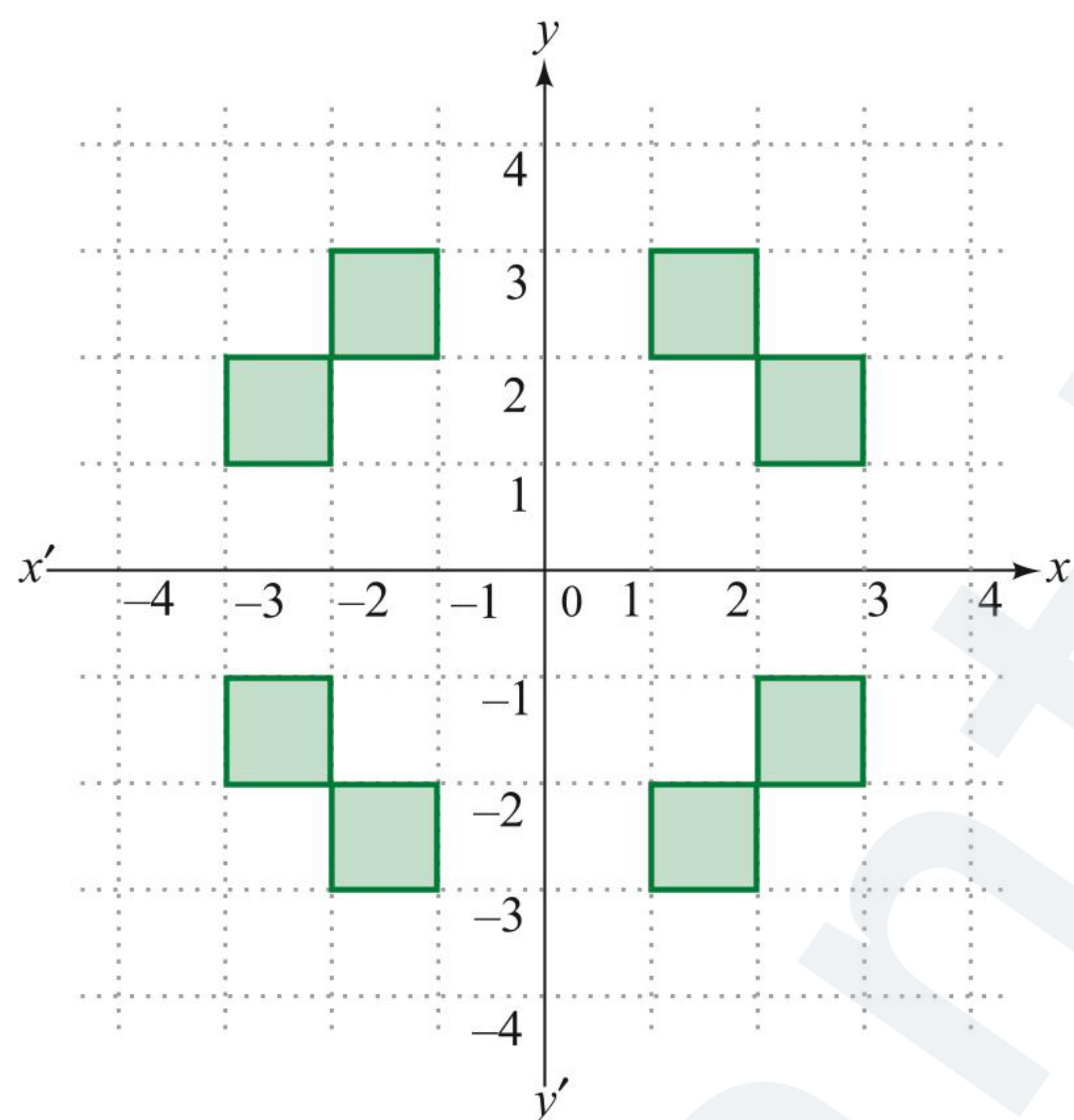


c. $\lfloor x \rfloor \lfloor y \rfloor = 2$

The graph is symmetrical about both x -axis and y -axis.

For $x, y > 0$; $\lfloor x \rfloor \lfloor y \rfloor = 2$

$\Rightarrow \lfloor x \rfloor = 1$ and $\lfloor y \rfloor = 2$ or $\lfloor x \rfloor = 2$ and $\lfloor y \rfloor = 1$.

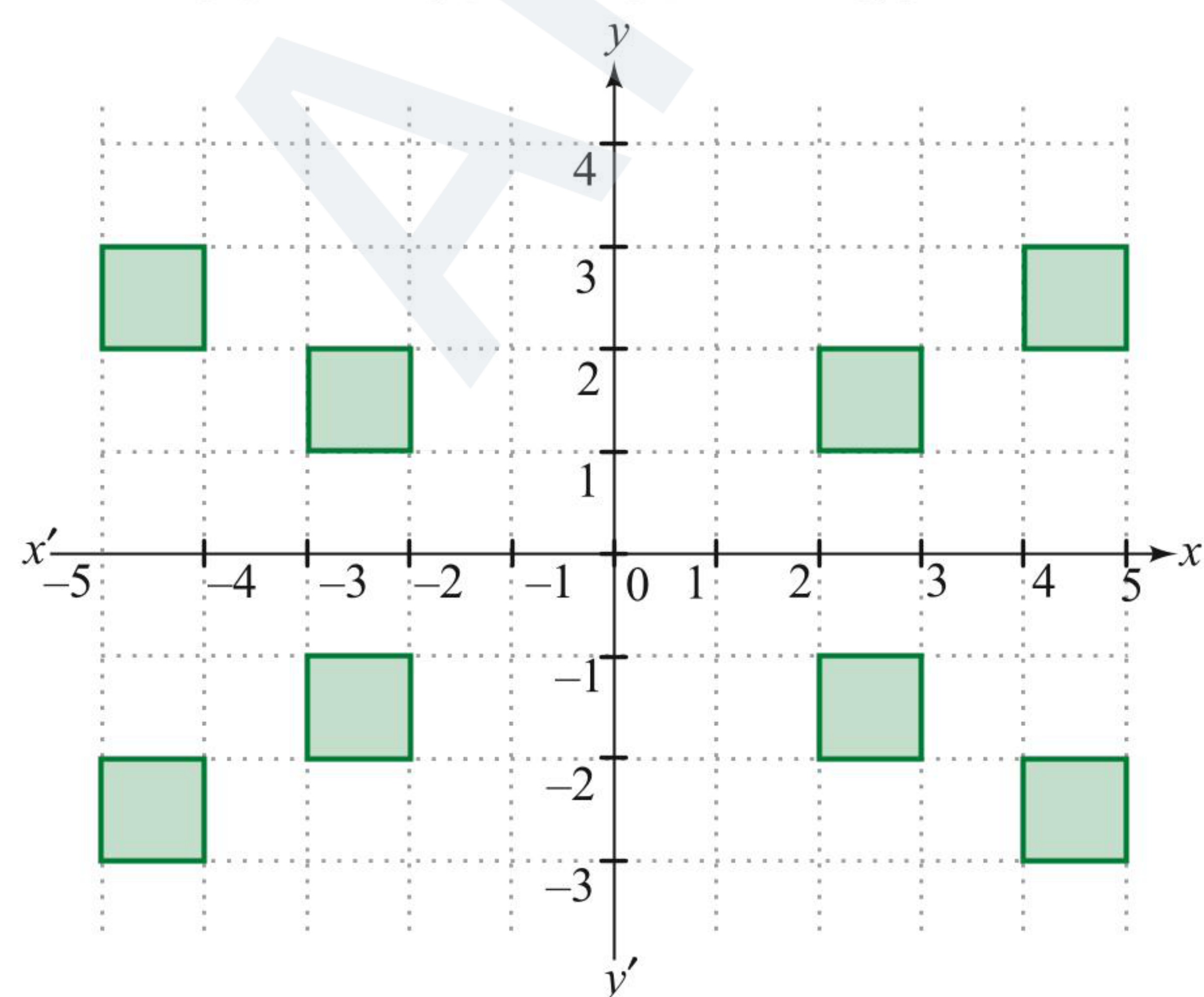


d. $\frac{\lfloor x \rfloor}{\lfloor y \rfloor} = 2$, where $-5 \leq x \leq 5$.

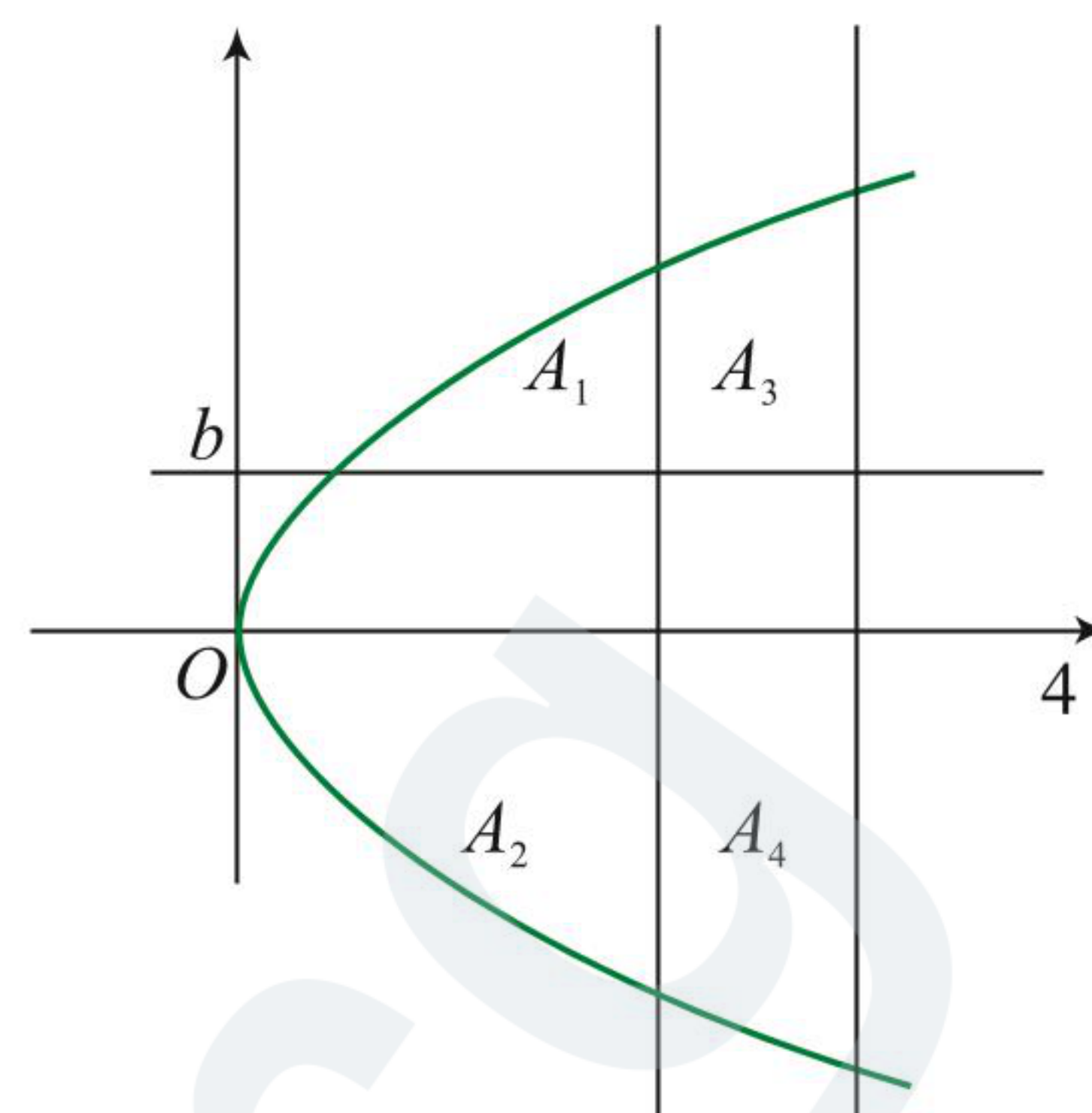
The graph is symmetrical about both the axes.

For $x, y > 0$, $\lfloor x \rfloor = 2\lfloor y \rfloor$, $\lfloor y \rfloor \neq 0$.

$\Rightarrow \lfloor x \rfloor = 2$ and $\lfloor y \rfloor = 1$ or $\lfloor x \rfloor = 4$ and $\lfloor y \rfloor = 2$.



4. (3) a.



$$\text{Total area} = 2 \int_0^4 \sqrt{x} dx = \frac{32}{3}$$

Area of each part = $8/3$

$$A_3 = A_4 \Rightarrow \int_a^4 (\sqrt{x} - b) dx = \int_a^4 (b + \sqrt{x}) dx = \frac{8}{3}$$

$$\therefore b = 0$$

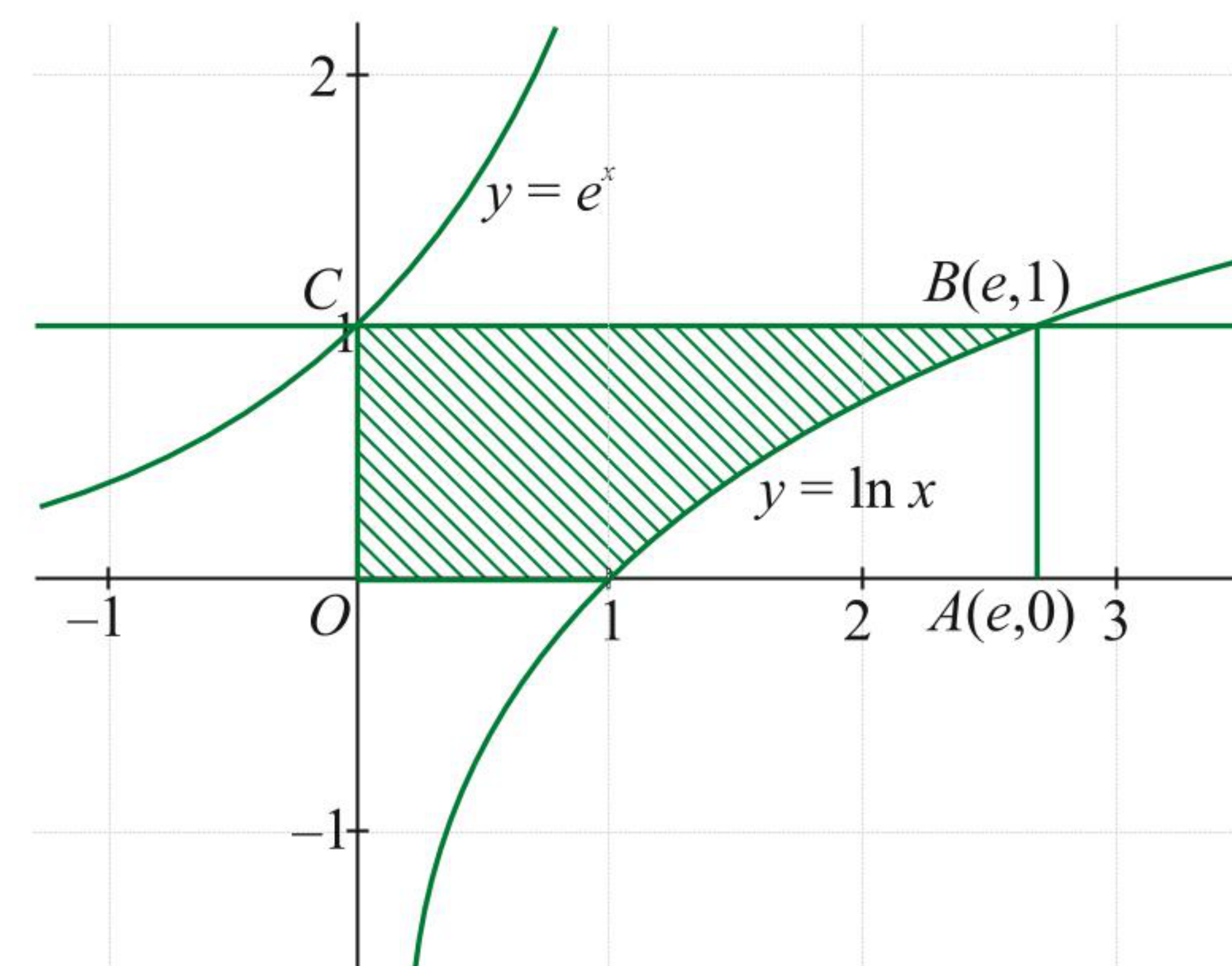
$$\therefore \int_a^4 \sqrt{x} dx = \frac{8}{3}$$

$$\therefore a^3 = 16$$

b. $f(x) = \begin{cases} x^{\frac{1}{\log_e x}}, & x \neq 1 \\ e, & x = 1 \end{cases} = \begin{cases} x^{\log_x e}, & x \neq 1 \\ e, & x = 1 \end{cases} = e$

Hence required area bound by $y = f(x)$ and $y = |x - e|$ is e^2

c.



$$\begin{aligned} \text{Area} &= \text{Area of rectangle } OABC - \int_1^e \ln x dx \\ &= e - 1 \text{ sq. units.} \end{aligned}$$

d. Solving $2 \cos x = 3 \tan x$ we get, $2 - 2 \sin^2 x = 3 \sin x$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$

$$\begin{aligned} \text{Required area} &= \int_0^{\pi/6} (2 \cos x - 3 \tan x) dx \\ &= (2 \sin x - 3 \ln \sec x) \Big|_0^{\pi/6} \\ &= 1 + \frac{3}{2} \log_e 3 - 3 \log_e 2 \end{aligned}$$

Numerical Value Type

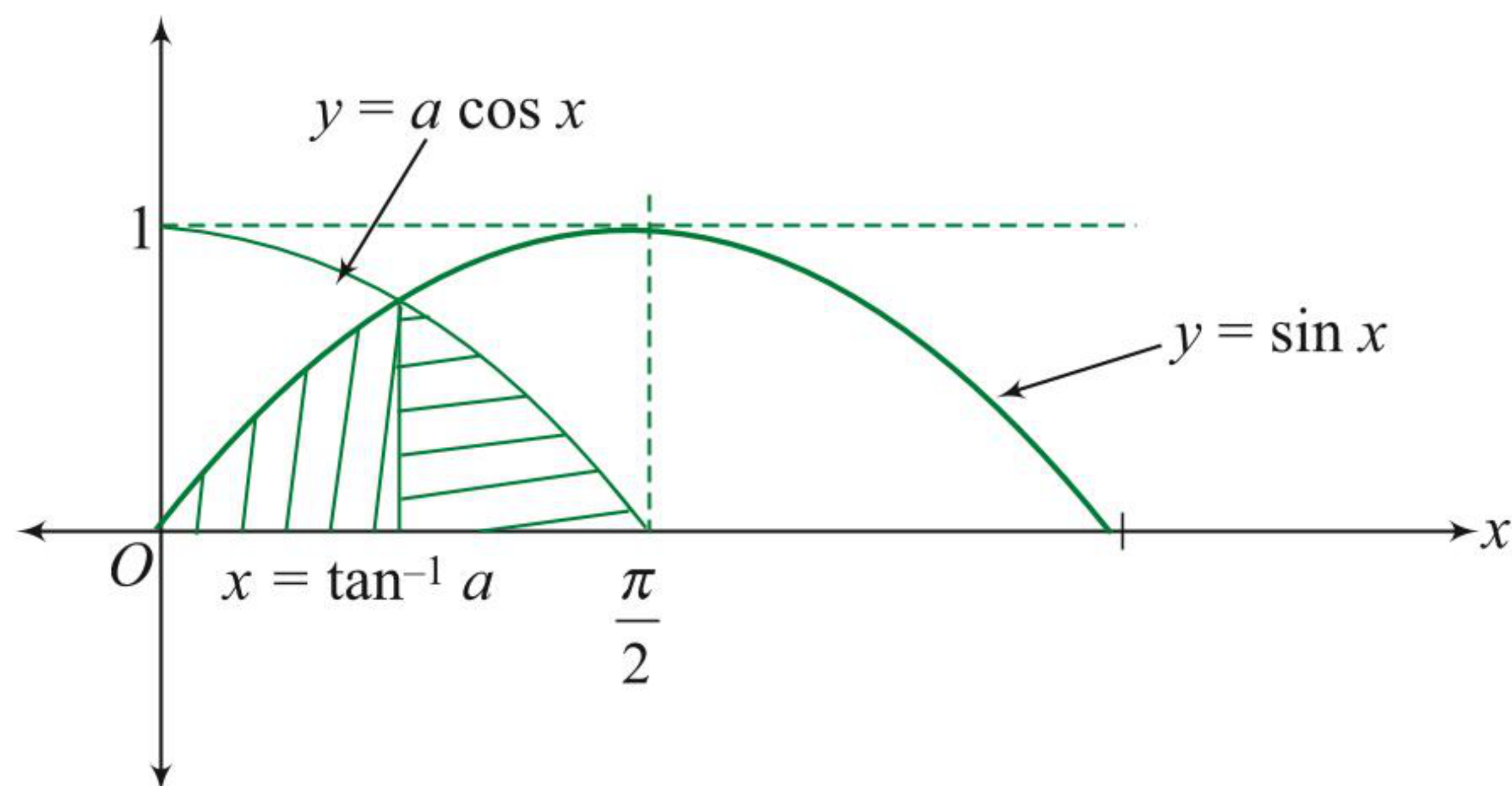
1. (9) Required area

$$A = \int_0^3 x \sqrt{9-x^2} dx; \text{ Put } 9-x^2 = t^2 \Rightarrow -2x dx = 2t dt$$

$$\therefore A = \int_0^3 t^2 dt = 9$$

2. (4) We have $S = \int_0^{\pi} \sin x dx = 2$, so $T = \frac{2}{3}$, where $a > 0$.

$$\text{Now } T = \int_0^{\tan^{-1} a} \sin x dx + \int_{\tan^{-1} a}^{\pi/2} a \cos x dx = \frac{2}{3}$$



$$\text{i.e., } -\cos(\tan^{-1} a) + 1 + a\{1 - \sin(\tan^{-1} a)\} = \frac{2}{3},$$

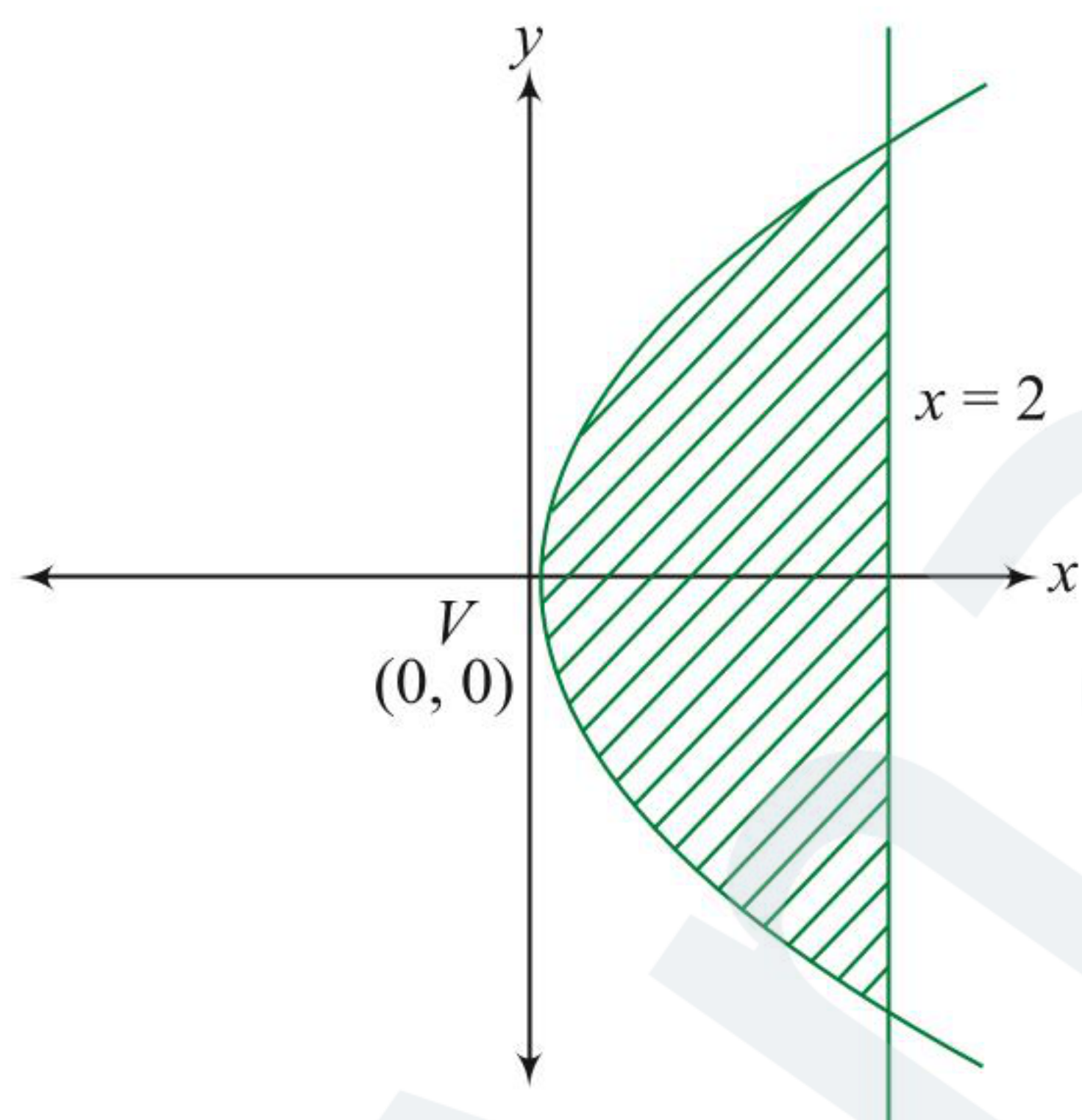
$$\text{i.e., } -\frac{1}{\sqrt{1+a^2}} + 1 + a - \frac{a^2}{\sqrt{1+a^2}} = \frac{2}{3}$$

$$\text{or } (a+1) - \sqrt{a^2+1} = \frac{2}{3} \text{ or } a + \frac{1}{3} = \sqrt{a^2+1}$$

$$\text{or } a = \frac{4}{3}$$

$$\text{Hence, } 3a = 4.$$

3. (8)

Let $P(x, y)$ be any point on the curve C .

$$\text{Now, } \frac{dy}{dx} = \frac{1}{y}$$

$$\text{or } y dy = dx \Rightarrow \frac{y^2}{2} = x + k$$

Since the curve passes through $M(2, 2)$, so $k = 0$

$$\Rightarrow y^2 = 2x$$

$$\therefore \text{ Required area} = 2 \int_0^2 \sqrt{2x} dx = 2\sqrt{2} \times \frac{2}{3} (x^{3/2})_0^2$$

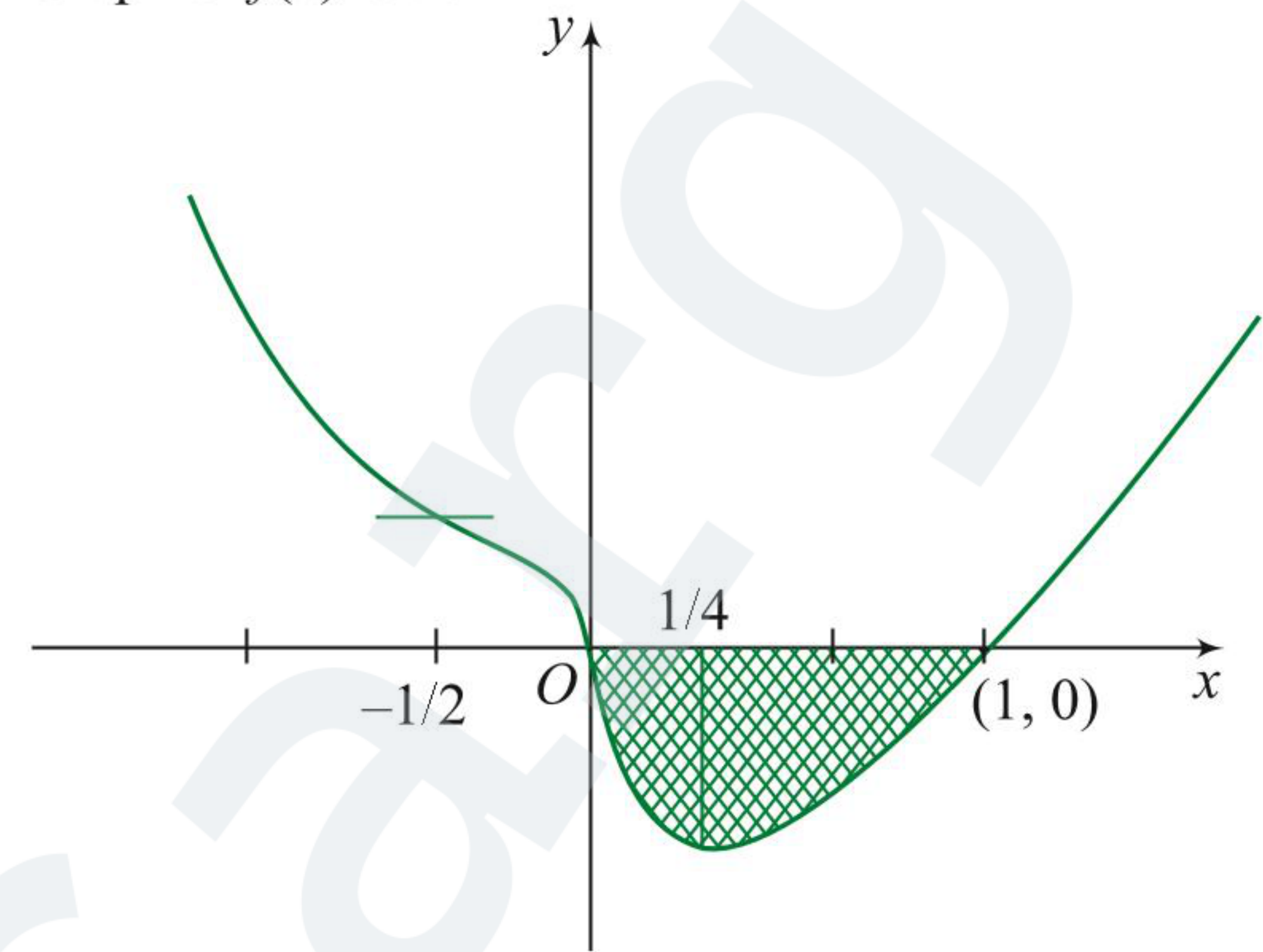
$$= \frac{4}{3} \sqrt{2} \times 2\sqrt{2} = \frac{16}{3} \text{ sq. unit}$$

$$4. (8) \int_0^3 (-x^2 + ax + 12) dx = 45 \text{ gives } a = 4$$

$$\text{Hence, } f(x) = 12 + 4x - x^2 = (2+x)(6-x)$$

$$\text{Hence, } m = -2 \text{ and } n = 6$$

$$m + n + a = 6 - 1 + 4 = 8$$

5. (9) Graph of $f(x)$ is as

$$A = \int_0^1 (x^{4/3} - x^{1/3}) dx = \left[\frac{3}{7} x^{7/3} - \frac{3}{4} x^{4/3} \right]_0^1$$

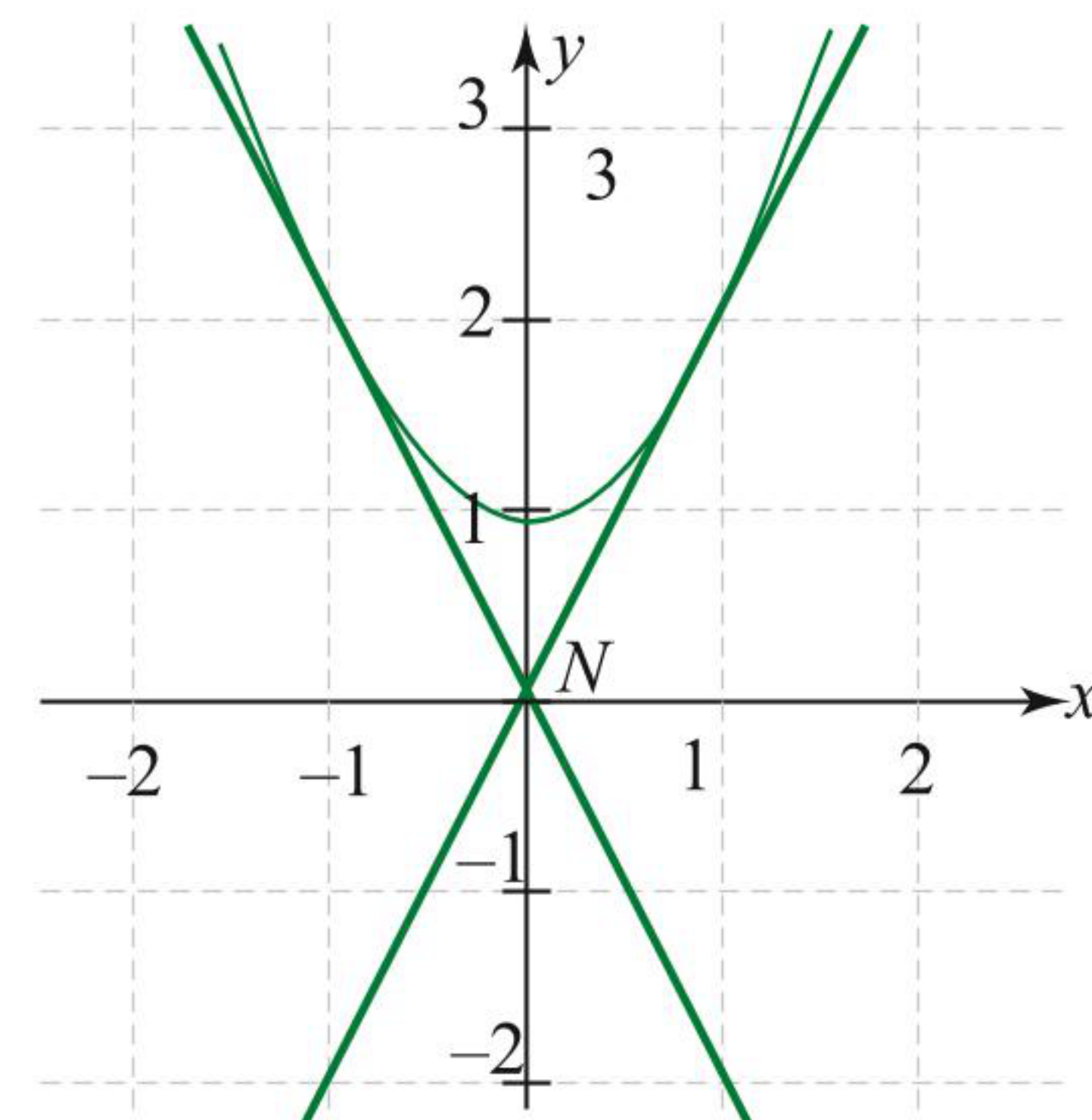
$$= \left| \frac{3}{7} - \frac{3}{4} \right| = 3 \left| \frac{4-7}{28} \right| = \frac{9}{28}$$

$$\text{or } 28A = 9$$

6. (2) Let the point of the curve be $(x, x^2 + 1)$.Now, the slope of tangent at this point is $2x$, which is equal to the slope of the line joining $(x, x^2 + 1)$ and $(0, 0)$.

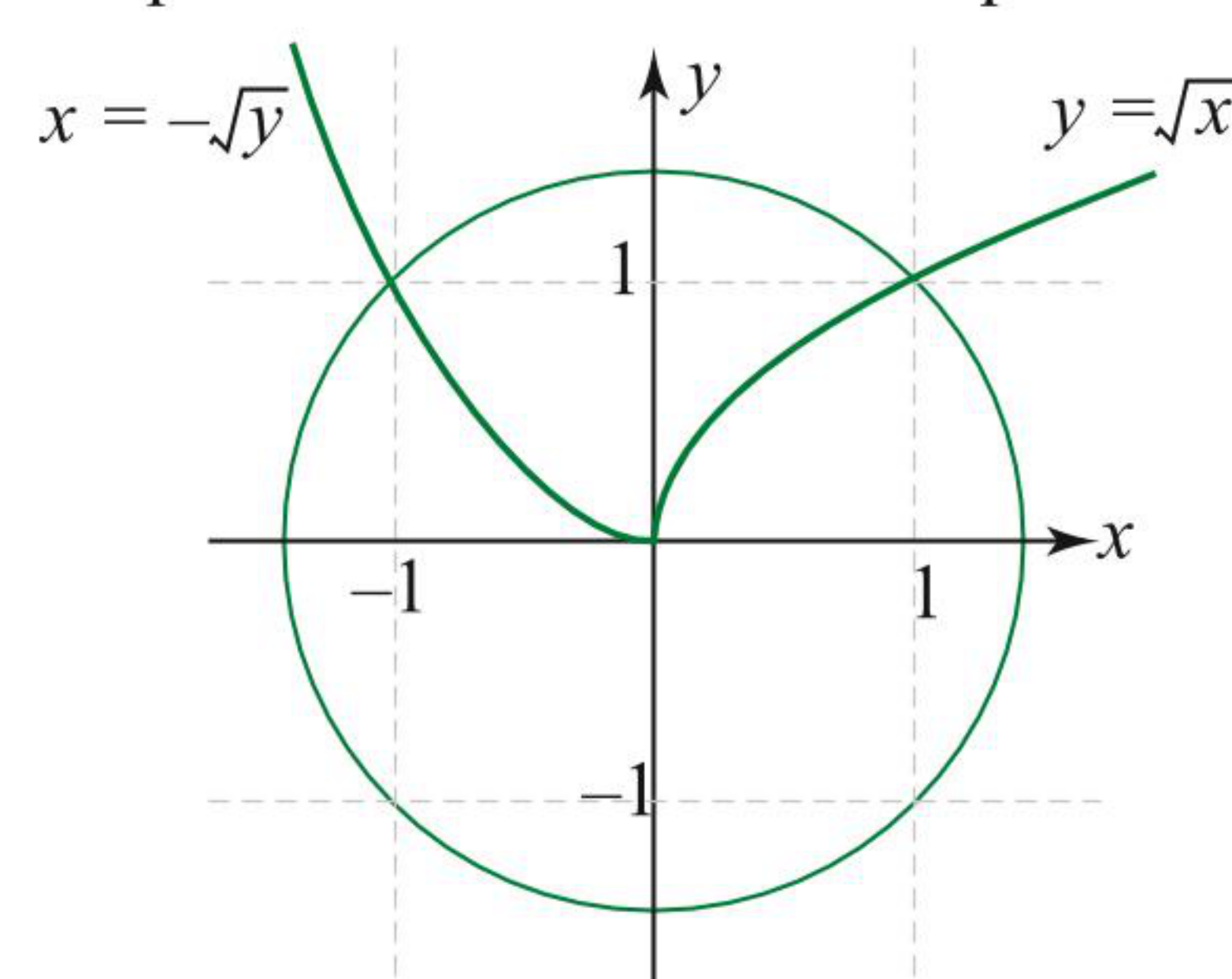
$$\text{Hence, } 2x = (x^2 + 1)/x \text{ or } 2x^2 = x^2 + 1$$

$$\text{or } x^2 = 1 \text{ or } x = \pm 1$$

Hence, equation of tangent is $y = \pm 2x$

$$\text{Now, Area} = 2 \int_0^1 (x^2 + 1 - 2x) dx = 2 \int_0^1 (x-1)^2 dx$$

$$= 2 \left[\frac{(x-1)^3}{3} \right]_0^1 = 2/3$$

7. (8) Required area = Area of one quadrant of the circle = $\pi/2$ 

$$8. (1) f(a) = \int_a^{2a} \left(\frac{x}{6} + \frac{1}{x^2} \right) dx = \left(\frac{x^2}{12} - \frac{1}{x} \right)_a^{2a}$$

$$= \left(\frac{4a^2}{12} - \frac{1}{2a} - \frac{a^2}{12} + \frac{1}{a} \right) = \frac{a^2}{4} + \frac{1}{2a}$$

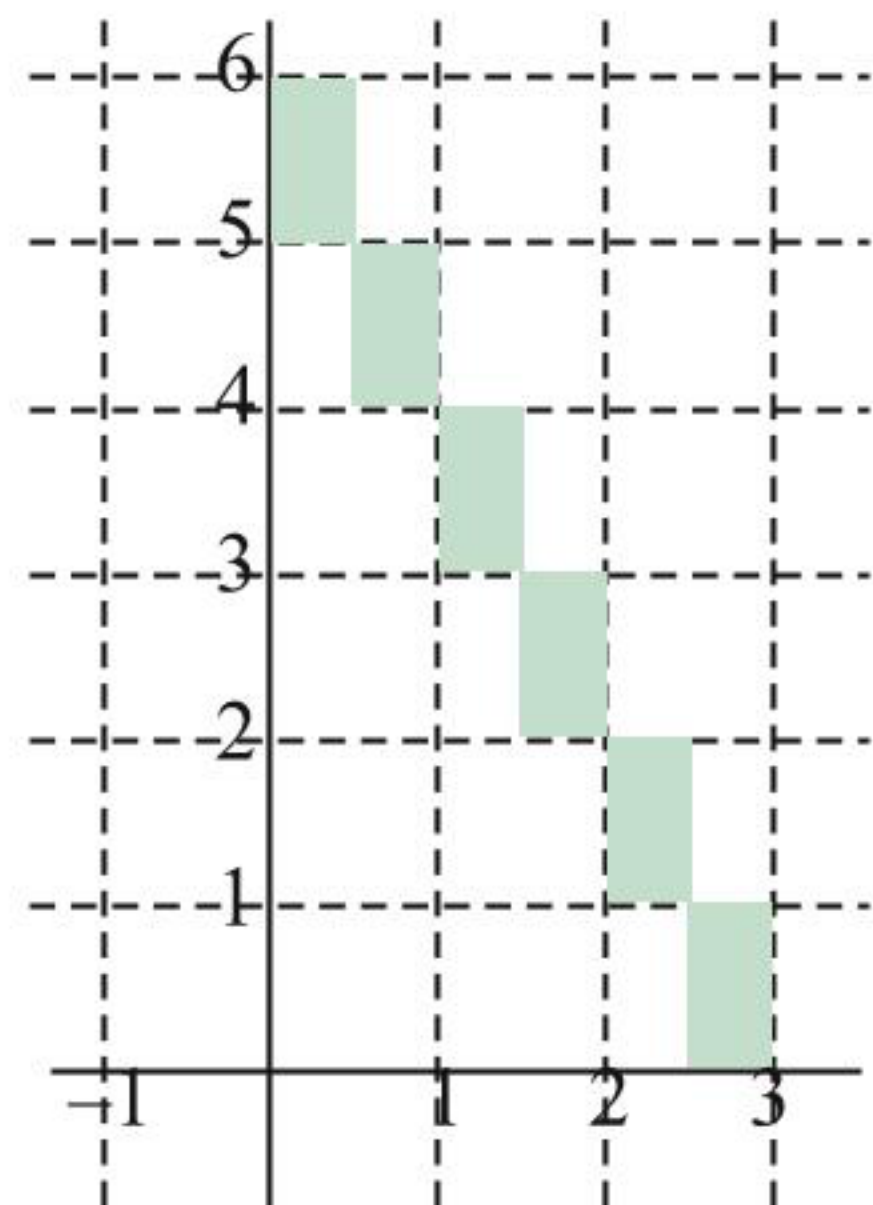
$$\text{Let } f'(a) = \frac{2a}{4} - \frac{1}{2a^2} = 0$$

$\Rightarrow a = 1$ which is point of minima.

$$9. (3) [2x] = 0 \Rightarrow 2x \in [0, 1)$$

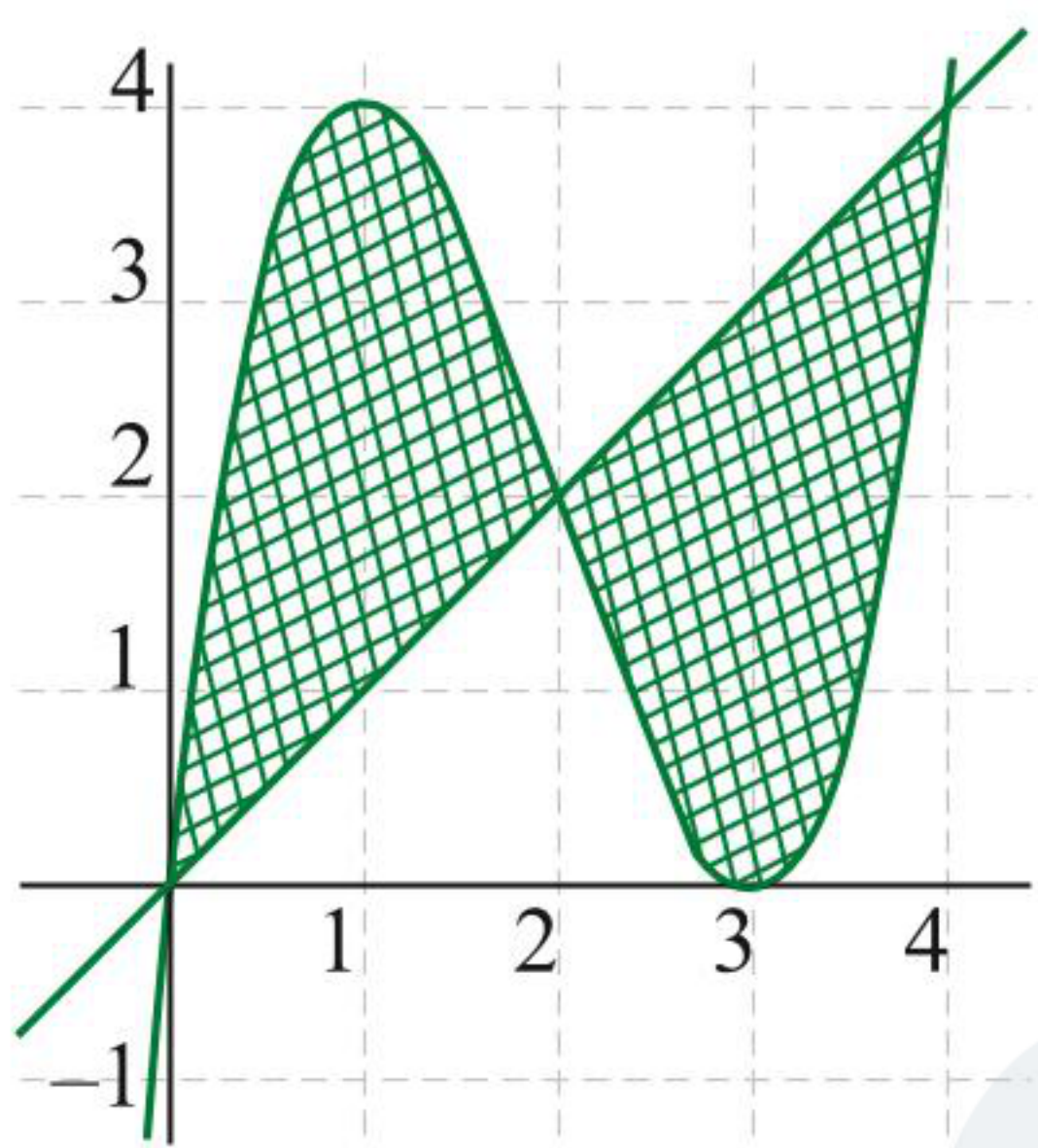
$$\Rightarrow x \in [0, 1/2) \Rightarrow [y] = 5 \Rightarrow y \in [5, 6)$$

Similarly we can consider $[2x] = 1, 2, 3, 4$ and 5



From the graph, area is 3 sq. units

$$10. (8) \text{ Required area} = 2 \int_0^2 (x(x-3)^2 - x) dx = 8 \text{ sq. units}$$

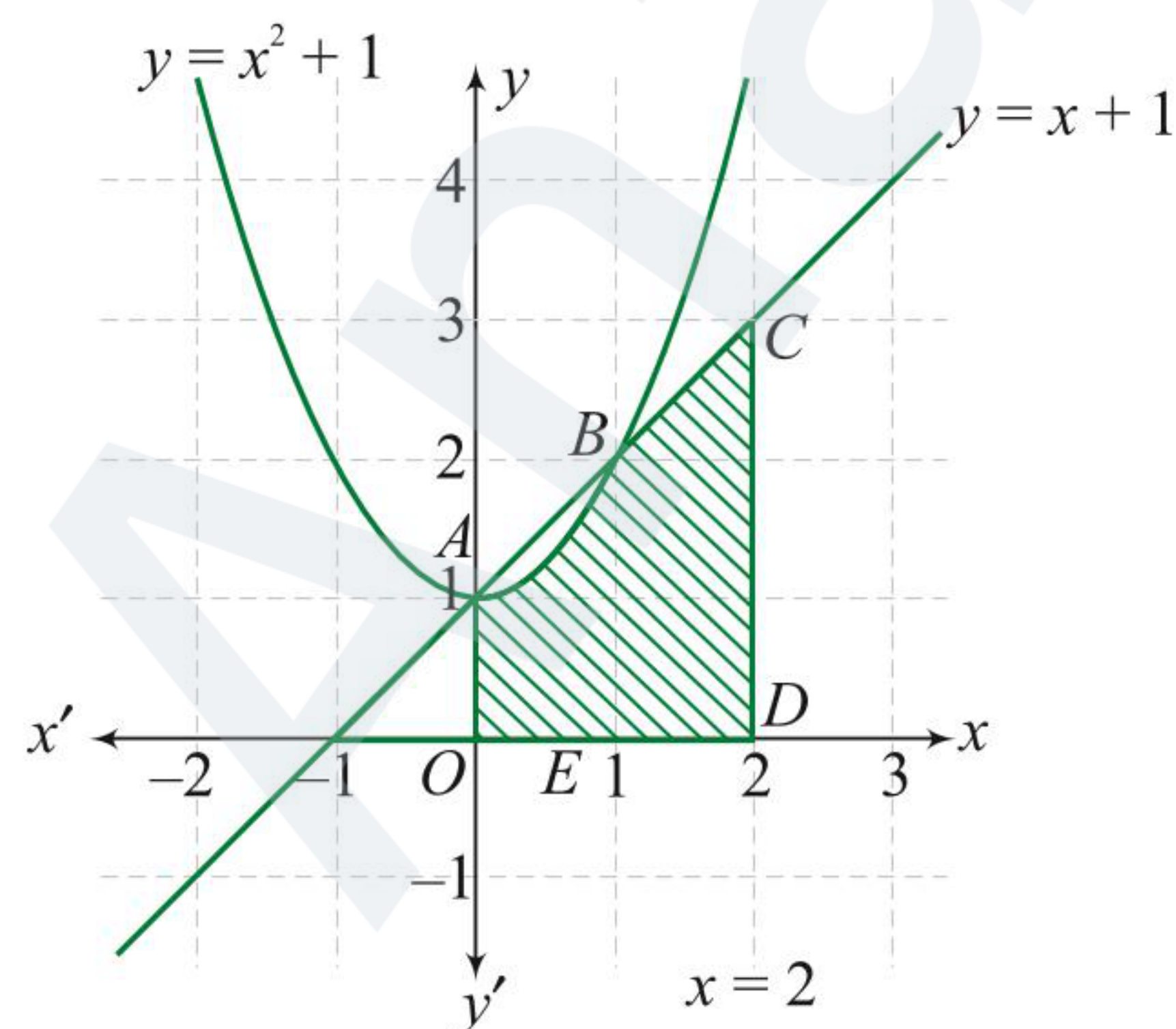


$$11. (6) \text{ Draw the given region point of intersection of } y = x^2 + 1$$

$$y = x + 1$$

$$x + 1 = x^2 + 1$$

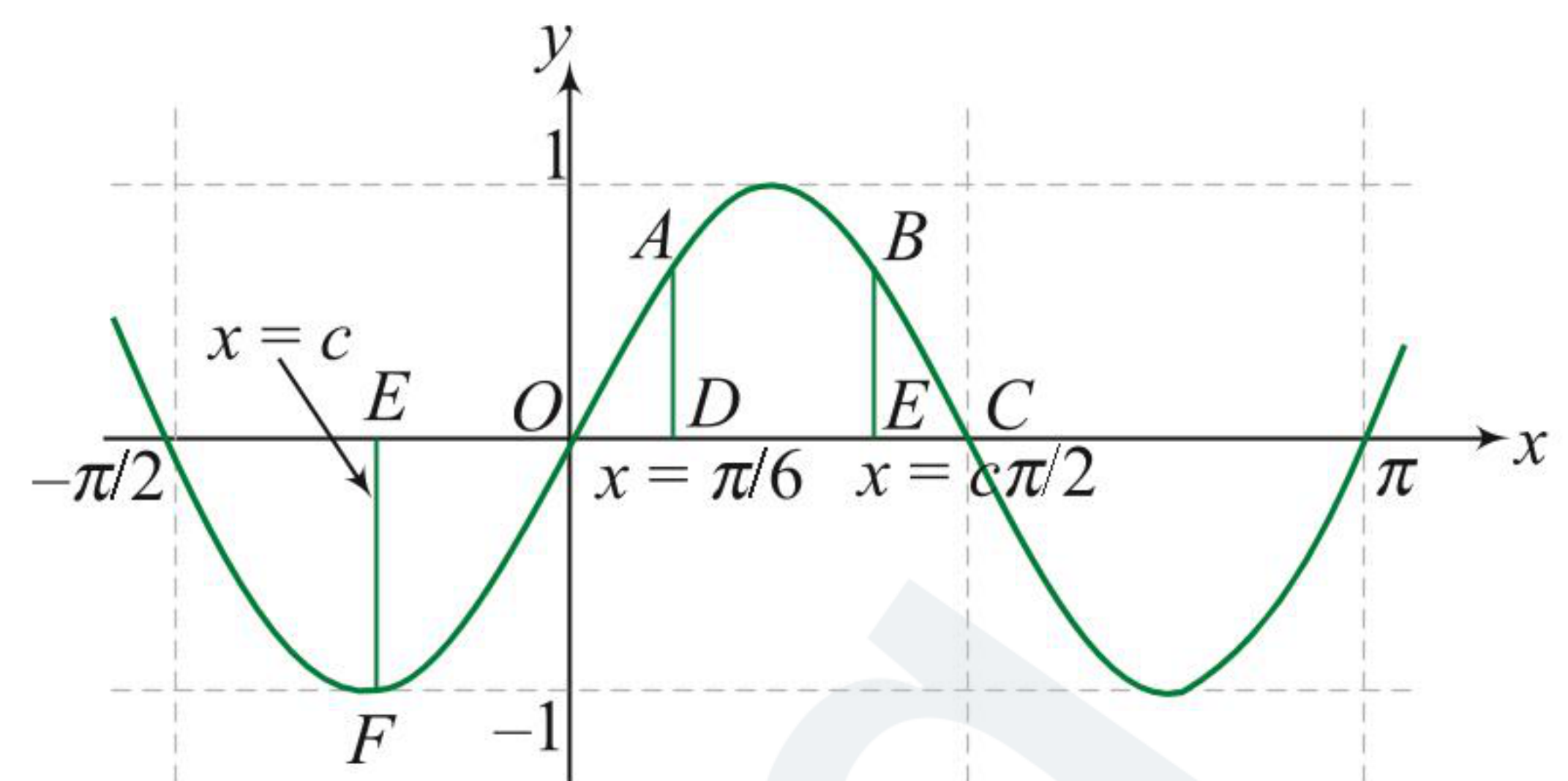
$$x = 0, 1$$



$$\text{Required area } OABCDE = \int_0^1 (x^2 + 1) dx + \int_1^2 (x + 1) dx$$

$$= \left(\frac{x^3}{3} + x \right)_0^1 + \left(\frac{x^2}{2} + x \right)_1^2 = \frac{23}{6} \text{ sq. units}$$

12. (6)



$$\text{Area } OABC = \int_0^{\pi/2} \sin 2x dx = 1$$

$$\text{Area } OAD = \int_0^{\pi/6} \sin 2x dx = \frac{1}{4}$$

Since $\sin 2x$ is symmetric about origin,

$$\text{so } c = -\frac{\pi}{6}, \text{ because area } OAD = \text{area } OEF$$

$$\int_{-\pi/6}^c \sin 2x dx = \frac{1}{2}$$

$$\cos 2c = -\frac{1}{2} \cos 2c = \frac{3}{2} \text{ (not possible)}$$

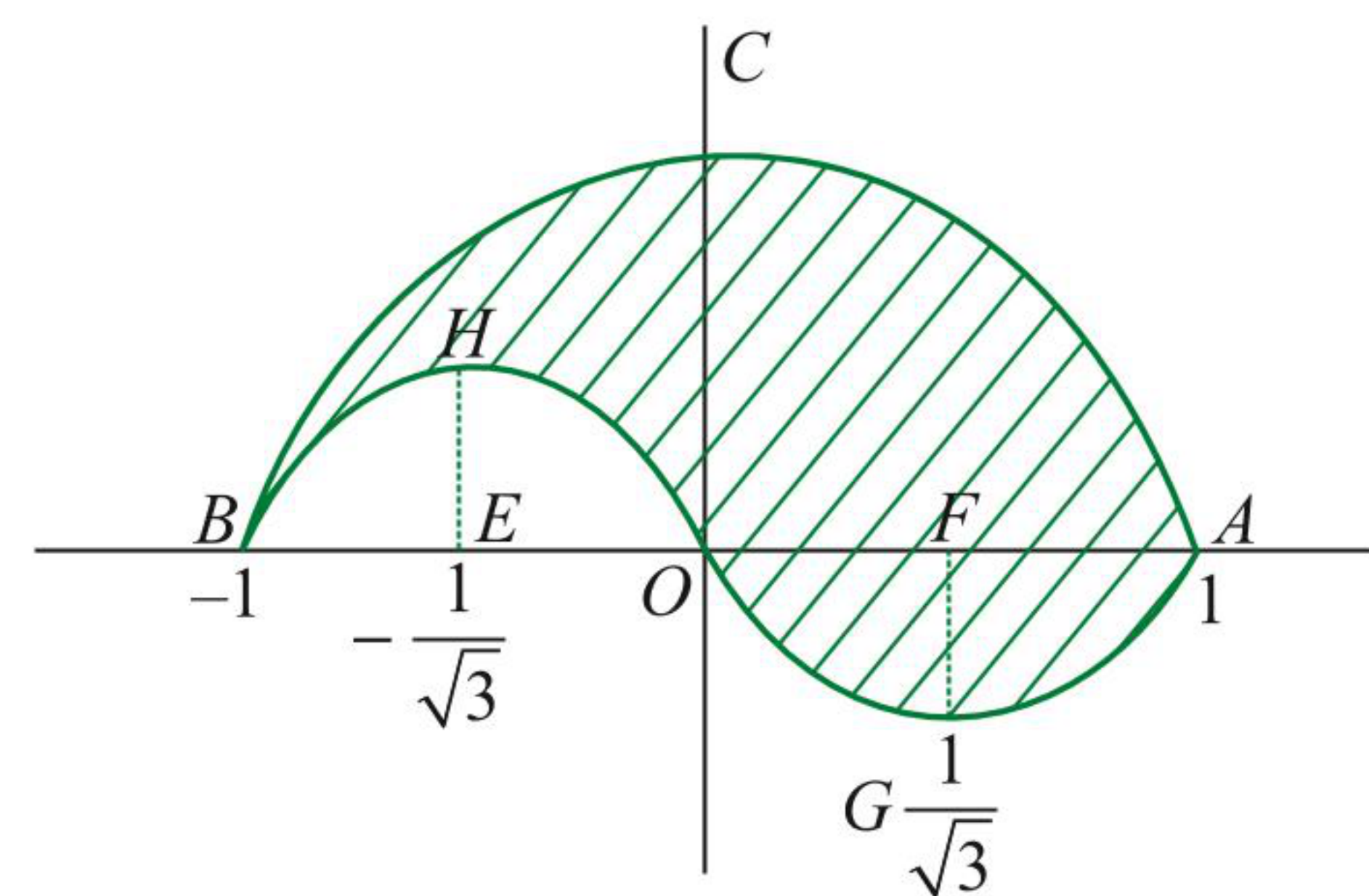
$$c = \frac{\pi}{3}$$

$$\text{so } c = -\frac{\pi}{6}, \frac{\pi}{3}$$

$$13. (2) y = \sqrt{1-x^2} \quad (1)$$

$$y = x^3 - x \quad (2)$$

$$y = 0 \text{ in (2) } x = 0, 1, -1$$



$$\text{Required area} = \text{Area of region } BCAGOH$$

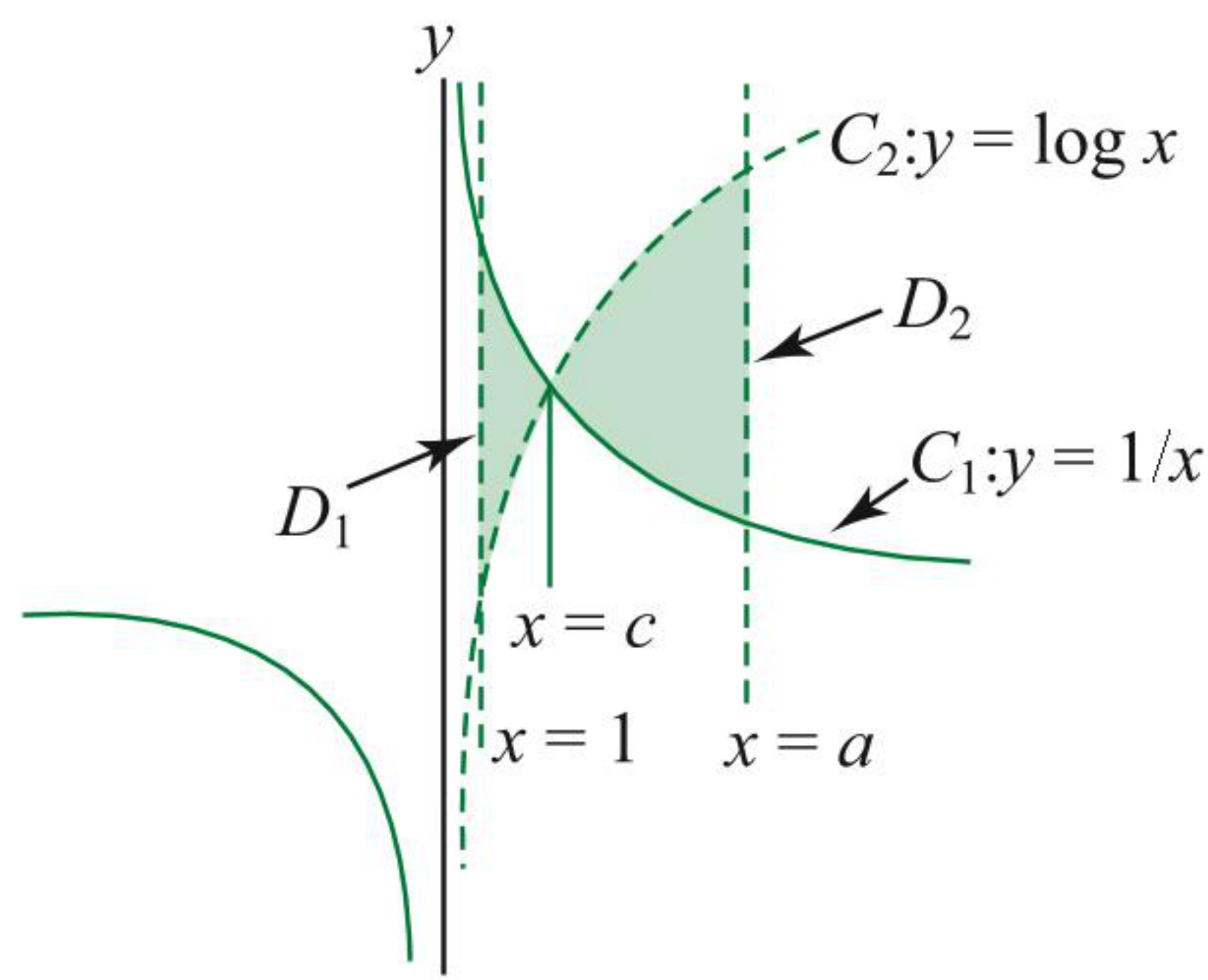
$$= \text{Area of semi-circle } BCAOB$$

$$= \frac{\pi}{2} \quad (\because \text{area of } BHOEB = \text{area of } OFAGO)$$

$$14. (1) \text{ Given that } D_1 = D_2$$

$$\int_1^c \left(\frac{1}{x} - \log x \right) dx = \int_c^a \left(\log x - \frac{1}{x} \right) dx$$

$$\left(\frac{-1}{x^2} - x(\log x - 1) \right)_1^c = \left(x(\log x - 1) + \frac{1}{x^2} \right)_c^a$$



$$\therefore 0 = a(\log a - 1) + \frac{1}{a^2}$$

$$\therefore a = 1$$

$$15. (3) \quad y = \frac{a^2 - ax}{1 + a^4} \quad (1)$$

$$= \frac{x^2 + 2ax + 3a^2}{1 + a^4} c \quad (2)$$

Point of intersection of (1) and (2)

$$\frac{a^2 - ax}{1 + a^4} = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$$

$$(x + a)(x + 2a) = 0$$

$$x = -a, -2a$$

$$\text{Req. area} = \int_{-2a}^{-a} \left[\left(\frac{a^2 - ax}{1 + a^4} \right) - \left(\frac{x^2 + 2ax + 3a^2}{1 + a^4} \right) \right] dx$$

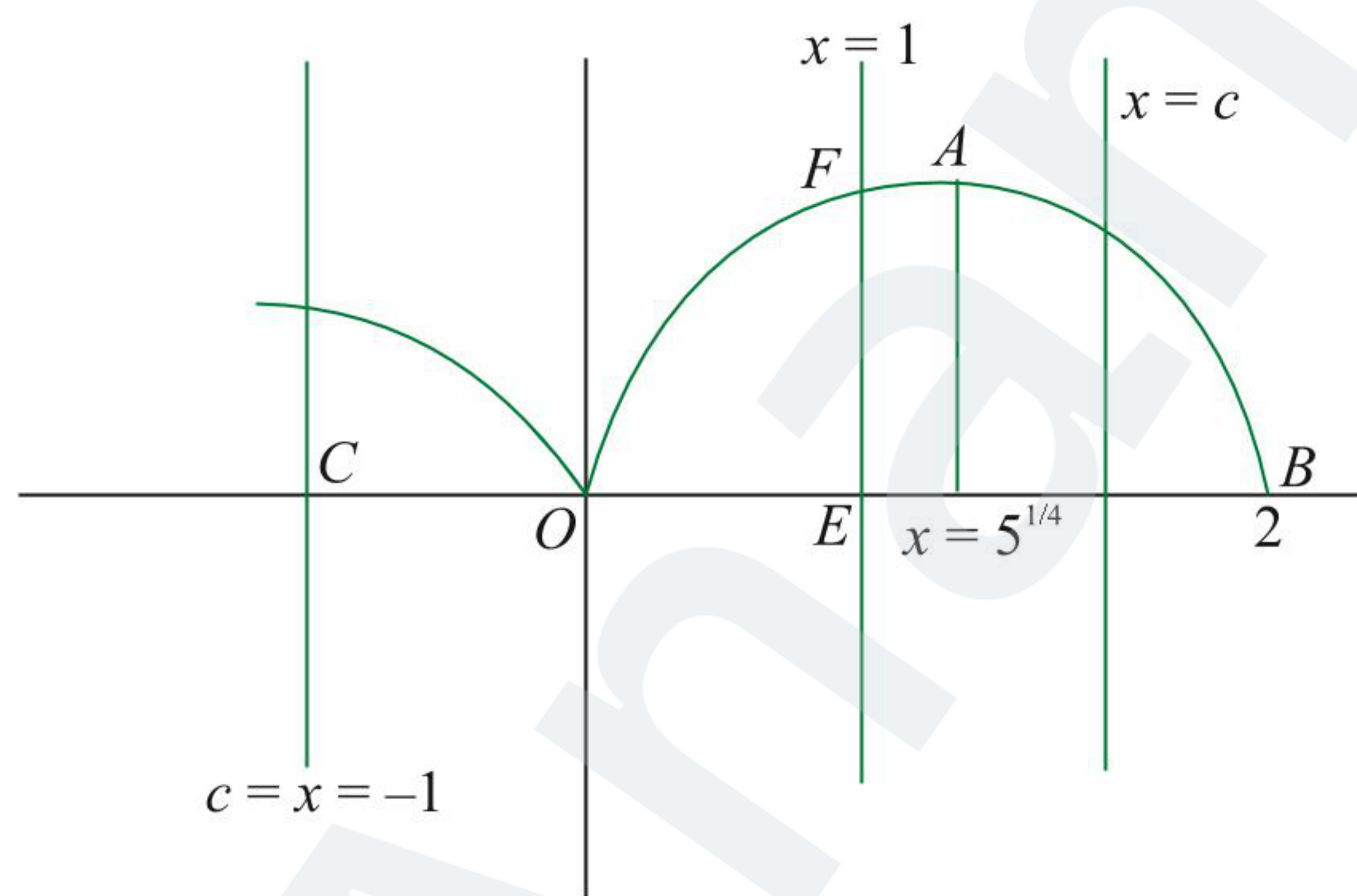
$$\therefore f(a) = \frac{a^3}{6(1 + a^4)}$$

If $f(a)$ is max, then $f'(a) = 0$

$$\Rightarrow 3 + 3a^4 - 4a^4 = 0$$

$$\text{or } a^4 = 3$$

16. (2)



$$\text{Given that } \int_1^c y dx = \frac{16}{3}$$

$$\Rightarrow \int_1^c (8x^2 - x^5) dx = \frac{16}{3} \quad c = (8 - \sqrt{17})^{1/3} \quad (c > 0)$$

$$\text{Area OFE} = \int_0^c (8x^2 - x^5) dx = \frac{8}{3} \quad (c > 0)$$

so $c = -1$

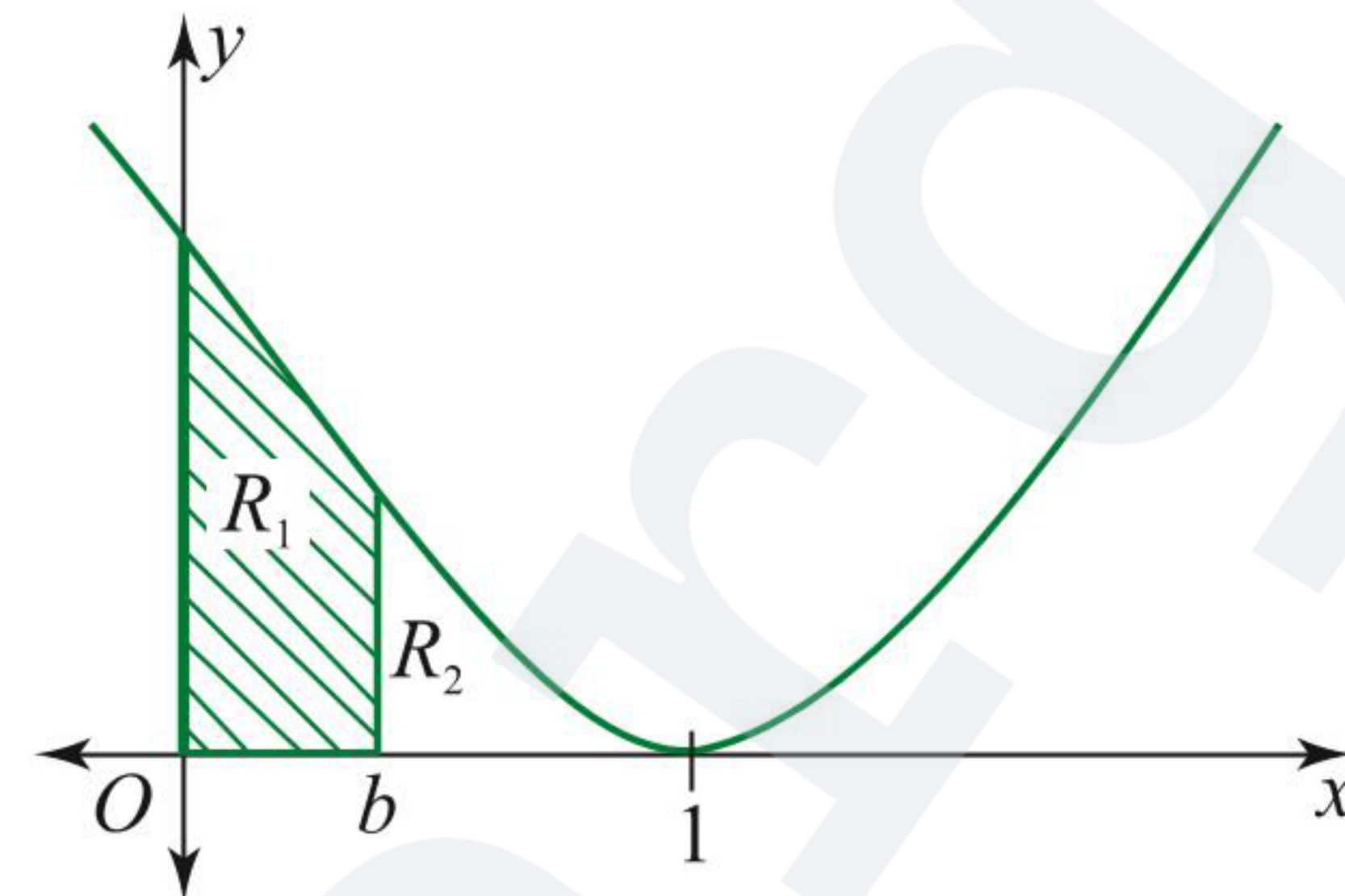
Hence, $c = -1$ and $(8 - \sqrt{17})^{1/3}$

Archives

JEE ADVANCED

Single Correct Answer Type

$$1. (2) \quad \therefore \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4}$$



$$\Rightarrow \frac{(x-1)^3}{3} \Big|_0^b - \frac{(x-1)^4}{4} \Big|_b^1 = \frac{1}{4}$$

$$\Rightarrow \frac{(b-1)^3}{3} + \frac{1}{3} - \left(0 - \frac{(b-1)^4}{4} \right) = \frac{1}{4}$$

$$\text{or } \frac{2(b-1)^3}{3} = -\frac{1}{12}$$

$$\text{or } (b-1)^3 = -\frac{1}{8} \quad \text{or } b = \frac{1}{2}$$

2. (2) Since $\sin x$ and $\cos x > 0$ for $x \in [0, \pi/2]$, the graph of $y = \sin x + \cos x$ always lies above the graph of $y = |\cos x - \sin x|$

Also $\cos x > \sin x$ for $x \in [0, \pi/4]$ and

$\sin x > \cos x$ for $x \in [\pi/4, \pi/2]$

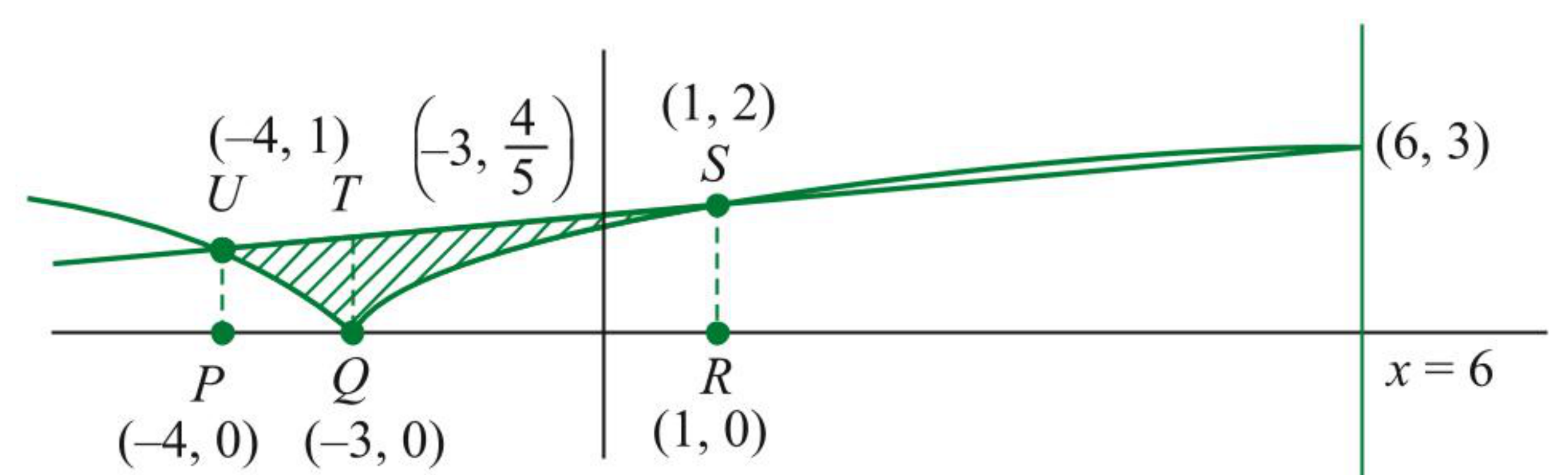
$$\Rightarrow \text{Area} = \int_0^{\pi/4} ((\sin x + \cos x) - (\cos x - \sin x)) dx$$

$$+ \int_{\pi/4}^{\pi/2} ((\sin x + \cos x) - (\sin x - \cos x)) dx$$

$$= 4 - 2\sqrt{2}$$

$$3. (3) \quad y \geq \sqrt{|x+3|}$$

$$y^2 \geq \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$$

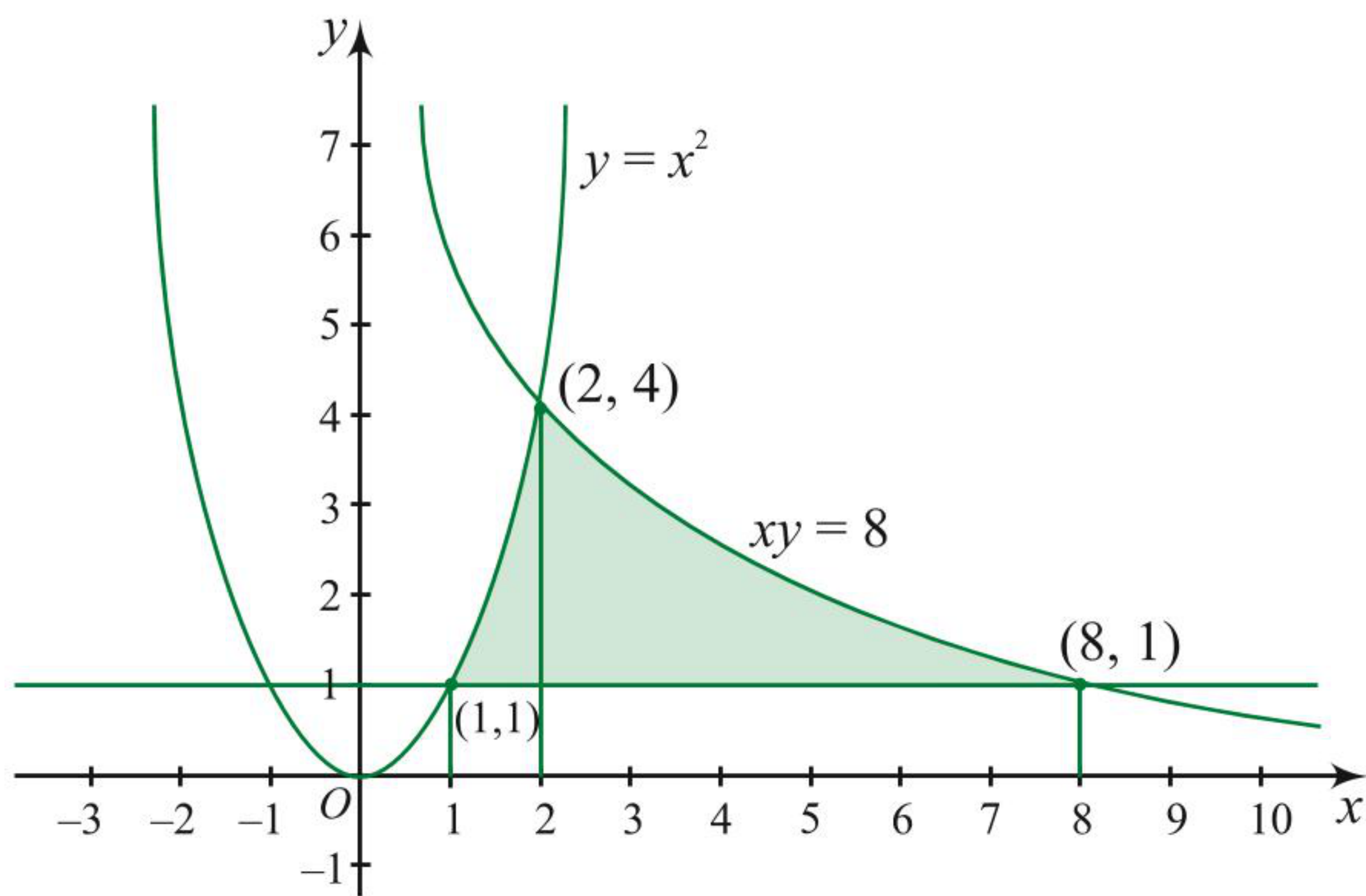


$$A = \left(A(\text{trapezium PQTU}) - \int_{-4}^{-3} \sqrt{-x-3} dx \right)$$

$$+ \left(A(\text{trapezium QRST}) - \int_{-3}^6 \sqrt{x+3} dx \right)$$

$$= \left(\frac{11}{10} - \frac{2}{3} \right) + \frac{16}{15} = \frac{3}{2}$$

4. (3)



$$\begin{aligned}
 \text{Required area} &= \int_1^2 x^2 dx + \int_2^8 \frac{8}{x} dx - 7 \\
 &= \left[\frac{x^3}{3} \right]_1^2 + 8[\log_e x]_2^8 - 7 \\
 &= \frac{7}{3} + 8[\log_e 8 - \log_e 2] - 7 \\
 &= 16\log_e 2 - \frac{14}{3}
 \end{aligned}$$

5. (1) We have

$$f(x) = \begin{cases} 0 & x < 1 \\ e^{x-1} - e^{1-x} & x \geq 1 \end{cases}$$

$$\text{and } g(x) = \frac{1}{2} (e^{x-1} + e^{1-x})$$

$$\text{Let } f(x) = g(x)$$

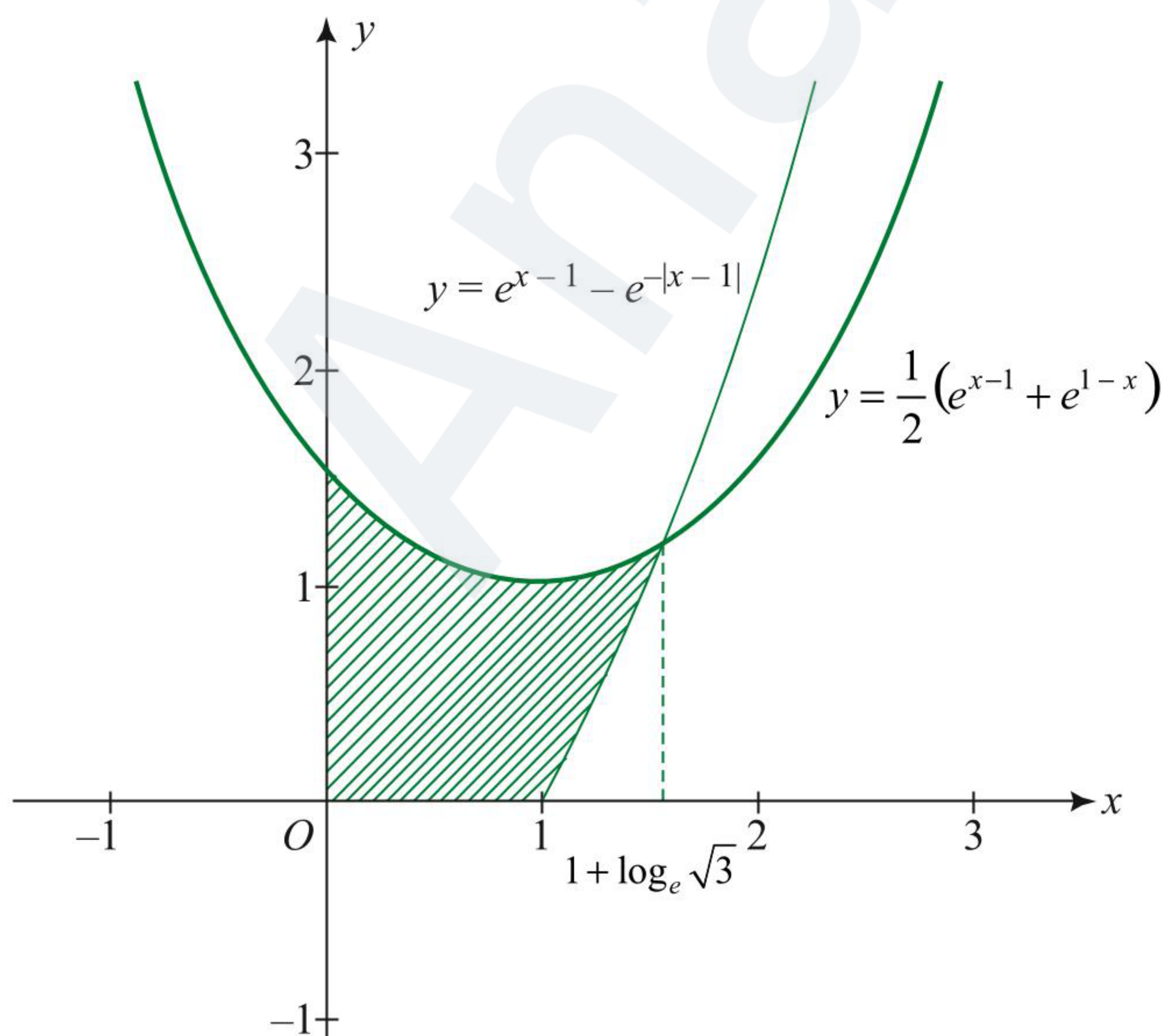
$$\therefore e^{x-1} - e^{1-x} = \frac{1}{2} (e^{x-1} + e^{1-x})$$

$$\Rightarrow e^{x-1} = 3e^{1-x}$$

$$\Rightarrow e^{2x} = 3e^2$$

$$\Rightarrow 2x = \log_e 3 + 2$$

$$\Rightarrow x = \log_e \sqrt{3} + 1$$

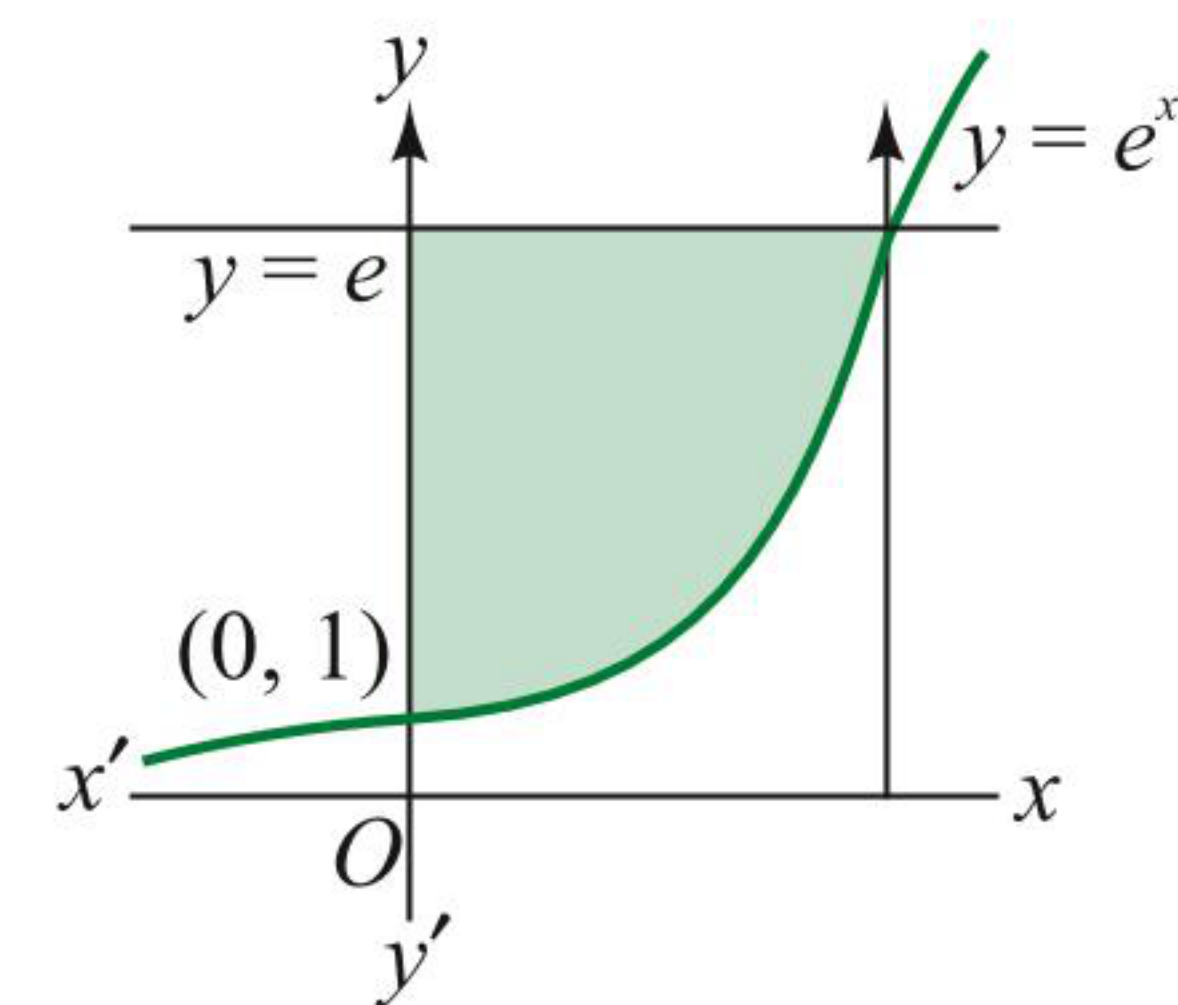
 $\therefore$ Required bounded area

$$\begin{aligned}
 &= \int_0^1 \frac{1}{2} (e^{x-1} + e^{1-x}) dx + \\
 &\quad \int_1^{1+\log_e \sqrt{3}} \left(\frac{1}{2} (e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x}) \right) dx \\
 &= \frac{1}{2} [e^{x-1} - e^{1-x}]_0^1 + \left[-\frac{1}{2} e^{x-1} - \frac{3}{2} e^{1-x} \right]_1^{1+\log_e \sqrt{3}} \\
 &= \frac{1}{2} \left[e - \frac{1}{e} \right] + \left[\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) + 2 \right] \\
 &= 2 - \sqrt{3} + \frac{1}{2} \left(e - \frac{1}{e} \right)
 \end{aligned}$$

Multiple Correct Answers Type

1. (2), (3), (4)

$$\begin{aligned}
 \text{Required area} &= \int_1^e \ln y dy \\
 &= (y \ln y - y)_1^e \\
 &= (e - e) - \{-1\} \\
 &= 1
 \end{aligned}$$



Also,

$$\int_1^e \ln y dy = \int_1^e \ln (e + 1 - y) dy$$

$$\text{Further Required area} = e \times 1 - \int_0^1 e^x dx$$

2. (1), (4)

Area between $y = x^3$ and $y = x$ in $x \in (0, 1)$ is

$$A = \int_0^1 (x - x^3) dx = \frac{1}{4}$$

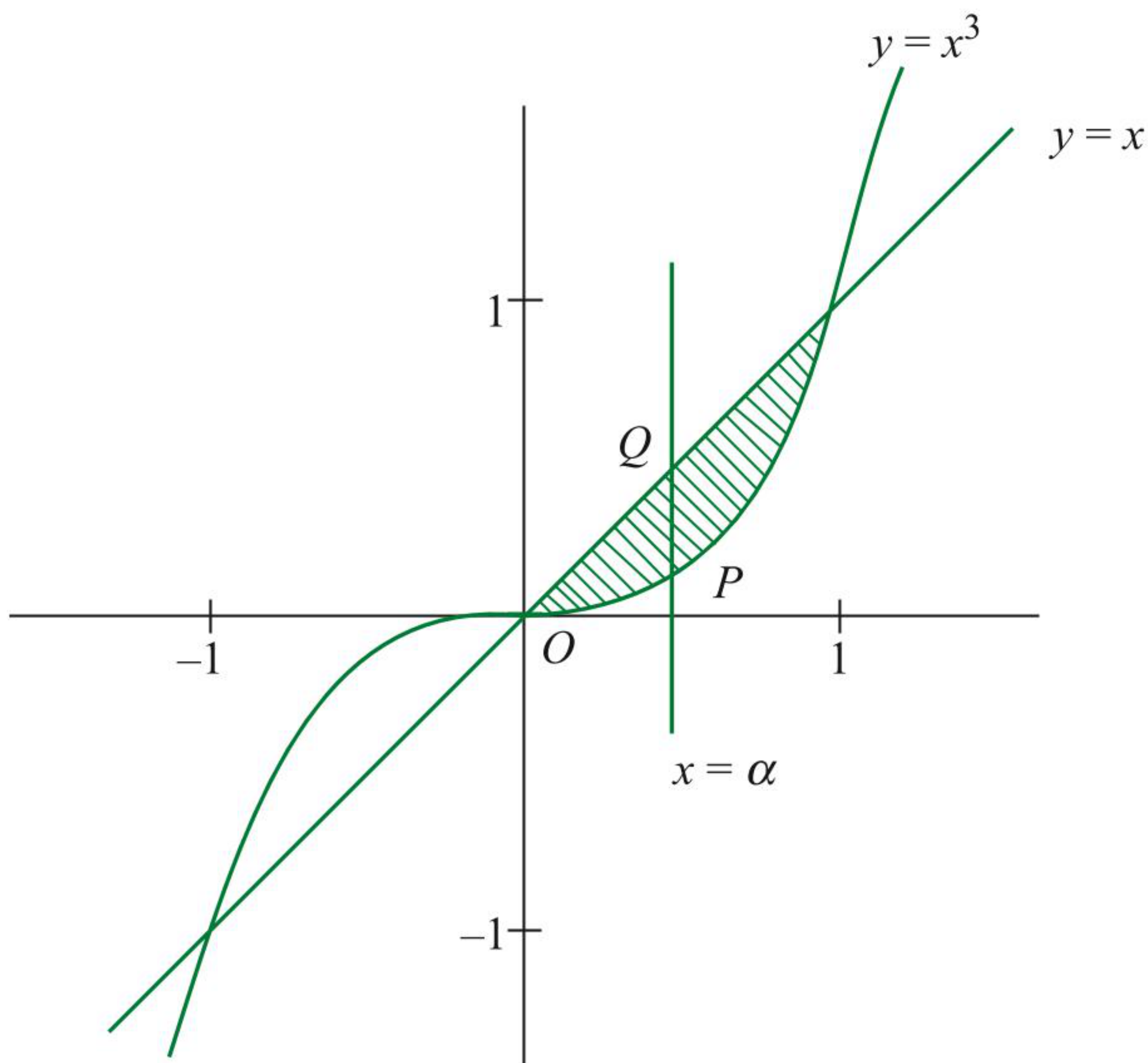
$$\text{Area of curvilinear triangle } OPQ = \frac{A}{2} = \frac{1}{8}$$

$$\Rightarrow \int_0^\alpha (x - x^3) dx = \frac{1}{8}$$

$$\Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$$

$$\Rightarrow (\alpha^2 - 1)^2 = \frac{1}{2}$$

$$\Rightarrow \alpha^2 = \frac{\sqrt{2} - 1}{\sqrt{2}}$$



3. (2), (3)

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t) dt$$

$$\Rightarrow f(x) = 1 - 2x + e^x \int_0^x e^{-t} f(t) dt \quad \dots(i)$$

Differentiating w.r.t. x , we get

$$f'(x) = -2 + e^x e^{-x} f(x) + e^x \int_0^x e^{-t} f(t) dt \quad \dots(ii)$$

Subtracting (i) from (ii), we get

$$\begin{aligned} f'(x) - f(x) &= -2 + f(x) - 1 + 2x \\ \Rightarrow f'(x) - 2f(x) &= 2x - 3 \end{aligned}$$

This is linear differential equation.

$$\text{I.F.} = e^{\int -2 dx} = e^{-2x}$$

Therefore, solution is

$$\begin{aligned} y \cdot e^{-2x} &= \int (2x - 3)e^{-2x} dx + c \\ \Rightarrow y \cdot e^{-2x} &= (2x - 3) \cdot \frac{e^{-2x}}{-2} - \int 2 \times \frac{e^{-2x}}{-2} dx + c \\ \Rightarrow y \cdot e^{-2x} &= -\frac{(2x - 3)e^{-2x}}{2} - \frac{e^{-2x}}{2} + c \\ \Rightarrow y \cdot e^{-2x} &= (1 - x)e^{-2x} + c \\ \Rightarrow y &= (1 - x) + c \cdot e^{2x} \quad \dots(iii) \end{aligned}$$

Putting $x = 0$ in (i), we get $f(0) = 1$.

So, from above equation, we have

$$1 = 1 + c \Rightarrow c = 0$$

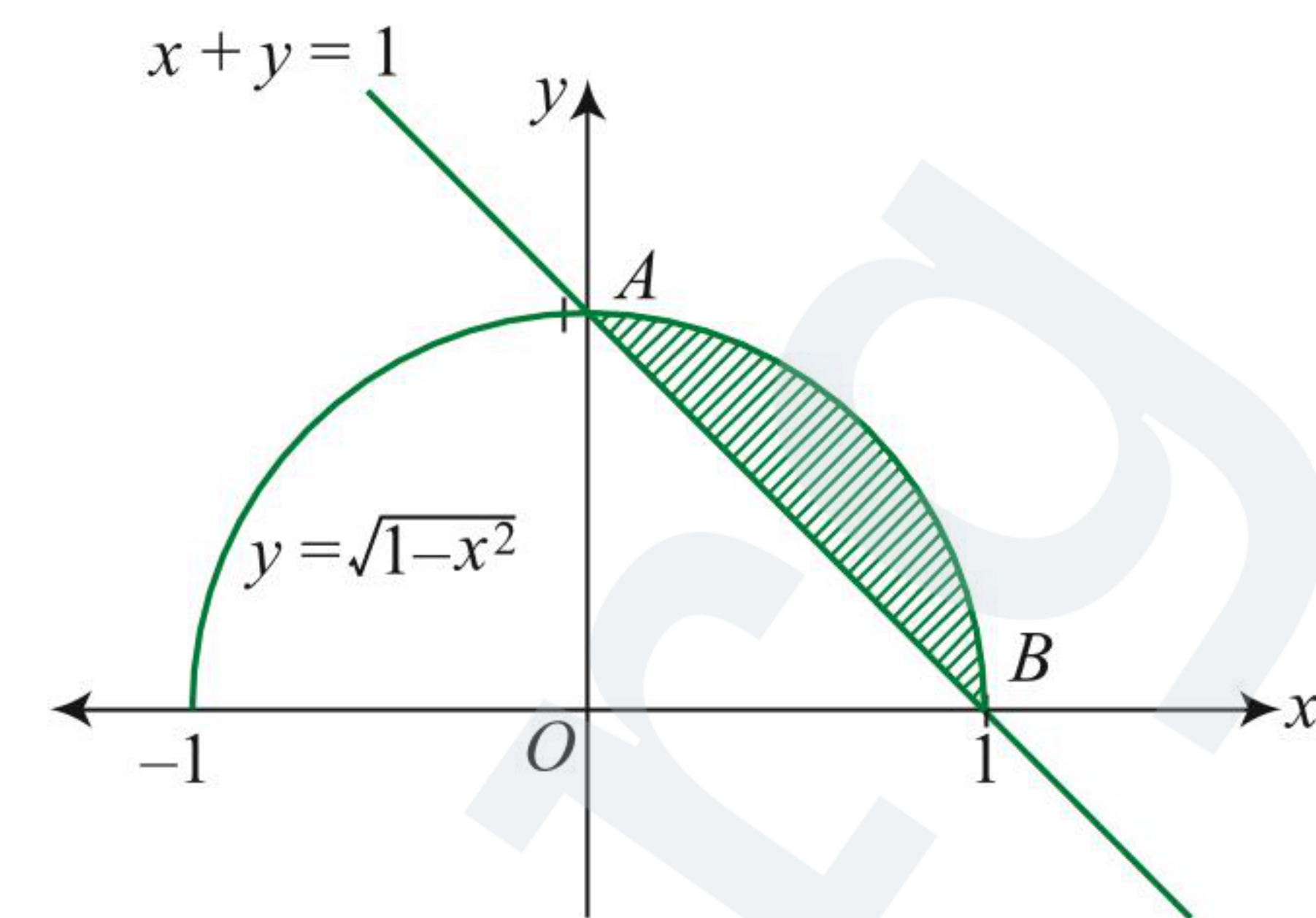
Now, (iii) reduces to $y = 1 - x$, which passes through point $(2, -1)$.

$$\text{Now, } f(x) \leq y \leq \sqrt{1 - x^2}$$

$$\Rightarrow 1 - x \leq y \leq \sqrt{1 - x^2}$$

$1 - x \leq y$ represents the region above the line $1 - x = y$ and $y \leq \sqrt{1 - x^2}$ represents the region inside the semicircle (lying above x -axis) $x^2 + y^2 = 1$.

So, common region is as the shaded one in the following figure:



Area of the shaded region

= Area of quarter of the circle - Area of triangle OAB

$$= \frac{1}{4} \times \pi(1)^2 - \frac{1}{2} \times 1 \times 1 = \frac{\pi - 2}{4}$$

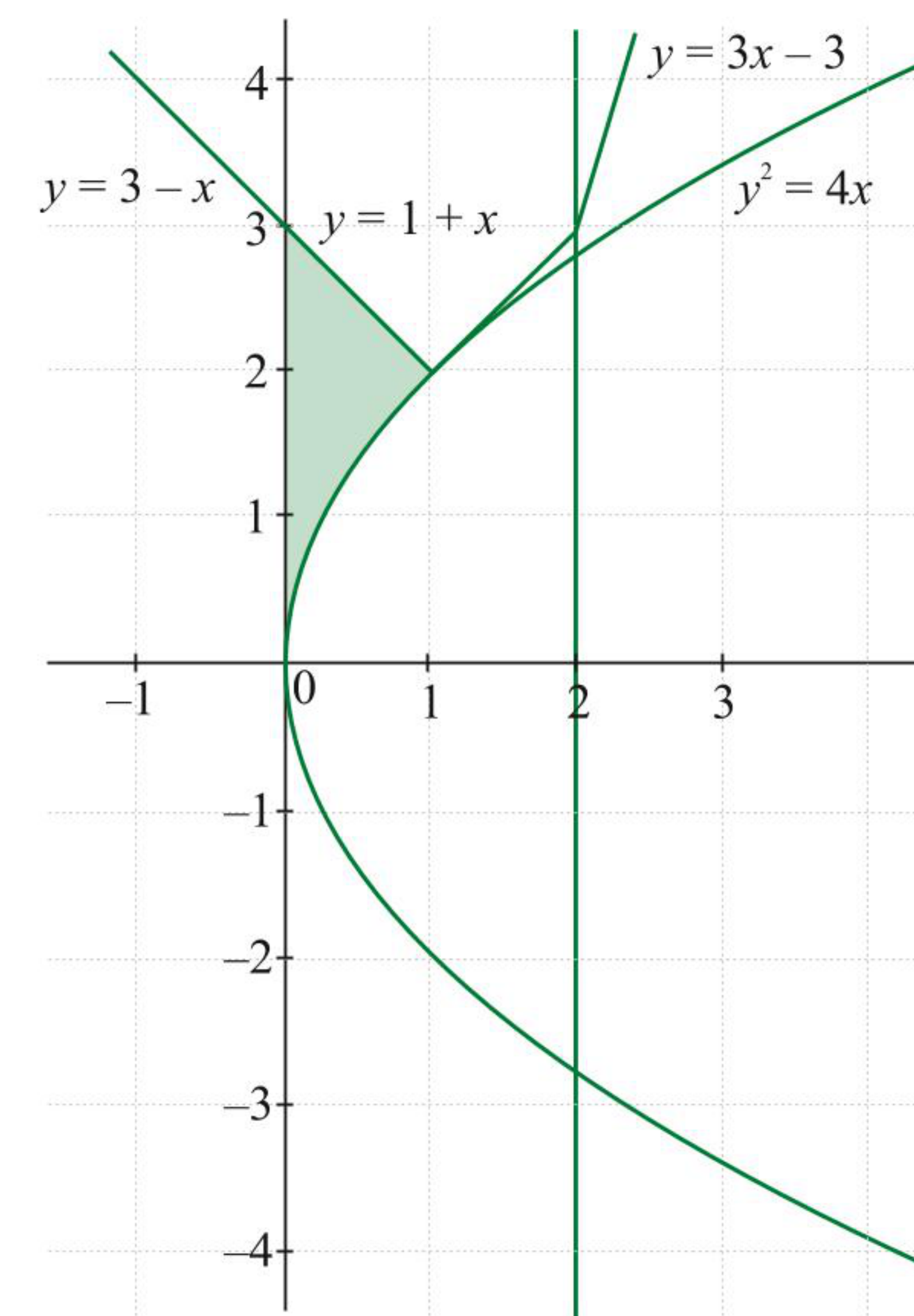
Matrix Match Type

1. $d \rightarrow s, t$

For $\alpha = 1$

$$y = |x - 1| + |x - 2| + x = \begin{cases} 3 - x; & x < 1 \\ 1 + x; & 1 \leq x < 2 \\ 3x - 3; & x \geq 2 \end{cases}$$

For $\alpha = 0, y = 3$

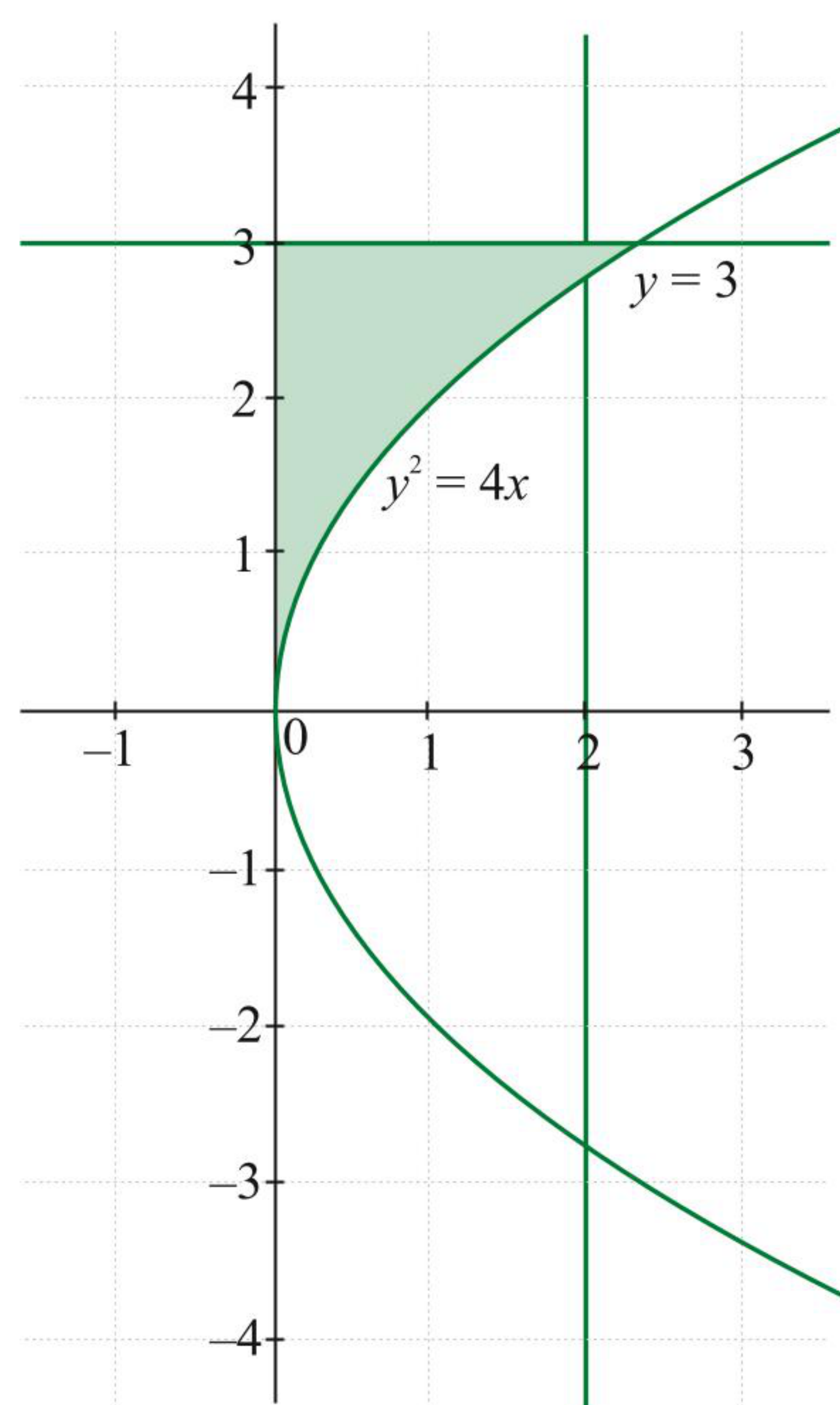


$$A = \frac{1}{2}(2 + 3) \times 1 + \frac{1}{2}(2 + 3) \times 1 - \int_0^2 2\sqrt{x} dx$$

$$\Rightarrow A = 5 - \frac{8}{3}\sqrt{2}$$

$$\therefore F(1) + \frac{8}{3}\sqrt{2} = 5$$

For $\alpha = 0, y = |-1| + |-2| = 3$



$$A = 6 - \int_0^2 2\sqrt{x} dx$$

$$\Rightarrow A = 6 - \frac{8}{3}\sqrt{2}$$

$$\therefore F(0) + \frac{8}{3}\sqrt{2} = 6$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (6) For $P(x, y)$, we have

$$2 \leq d_1(P) + d_2(P) \leq 4$$

$$\Rightarrow 2 \leq \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \leq 4$$

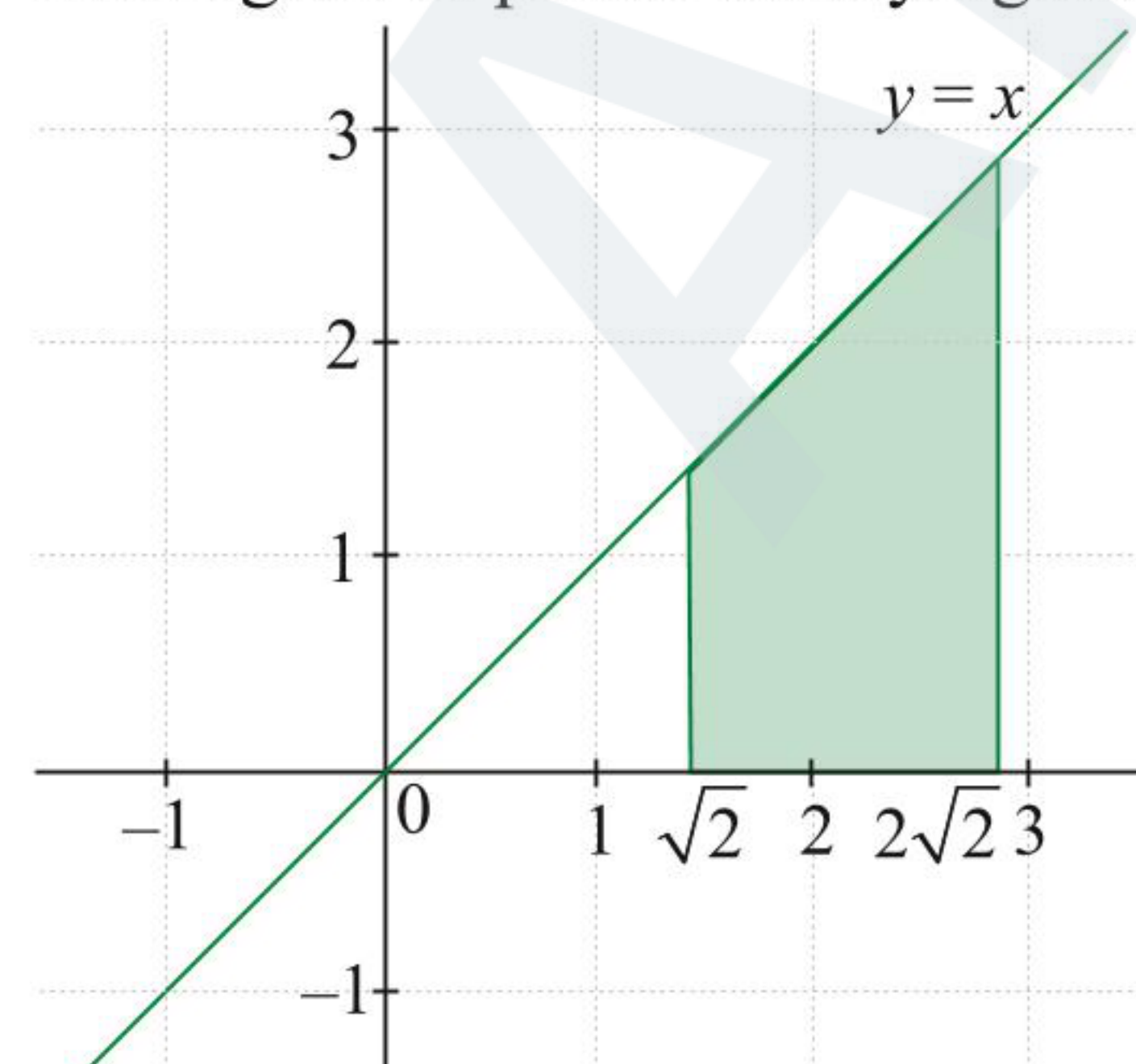
$$\Rightarrow 2\sqrt{2} \leq |x-y| + |x+y| \leq 4\sqrt{2}$$

In first quadrant if $x > y$, we have

$$2\sqrt{2} \leq x - y + x + y \leq 4\sqrt{2}$$

$$\text{or } \sqrt{2} \leq x \leq 2\sqrt{2}$$

The region of points satisfying these inequalities is

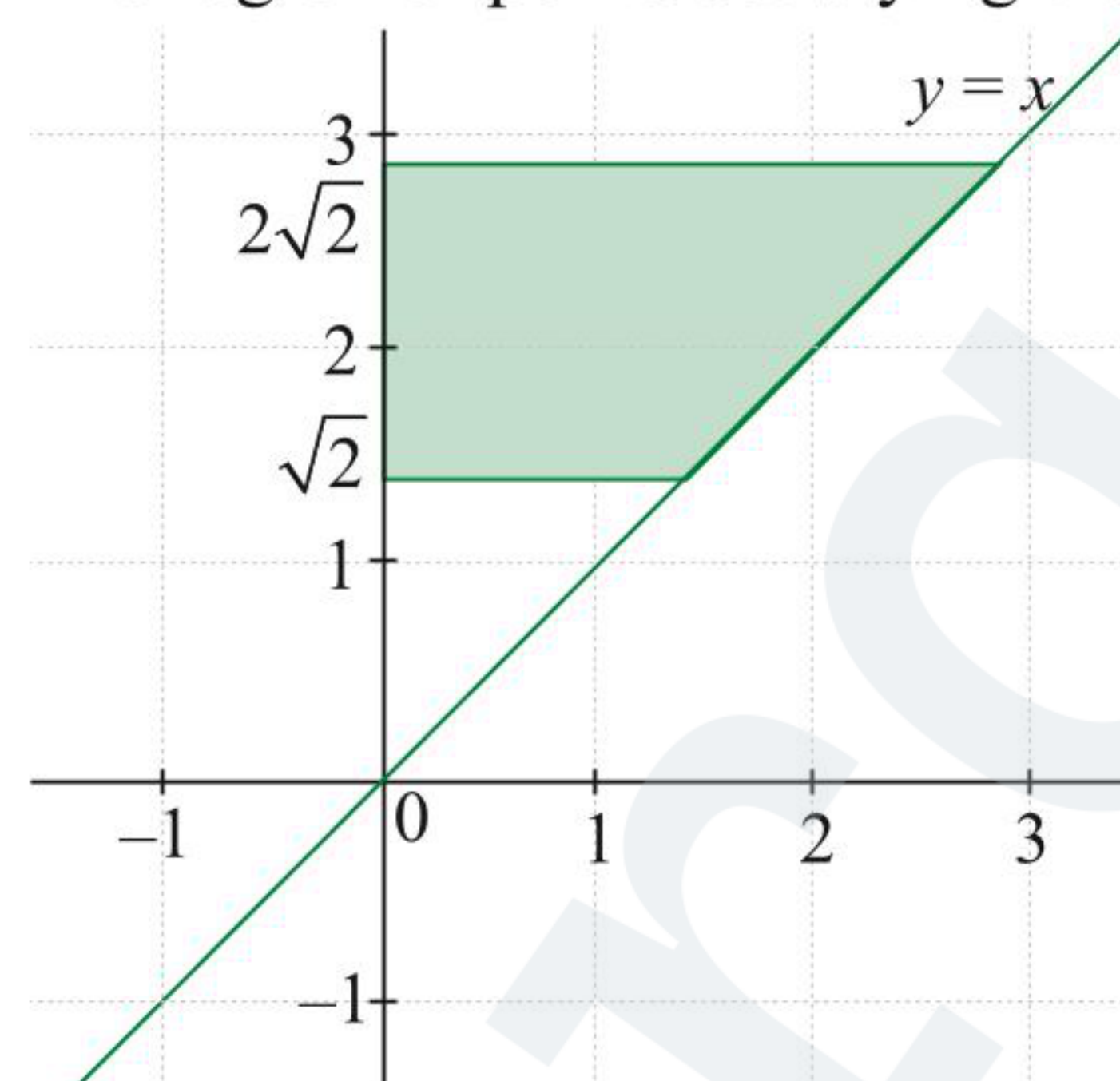


In first quadrant if $x < y$, we have

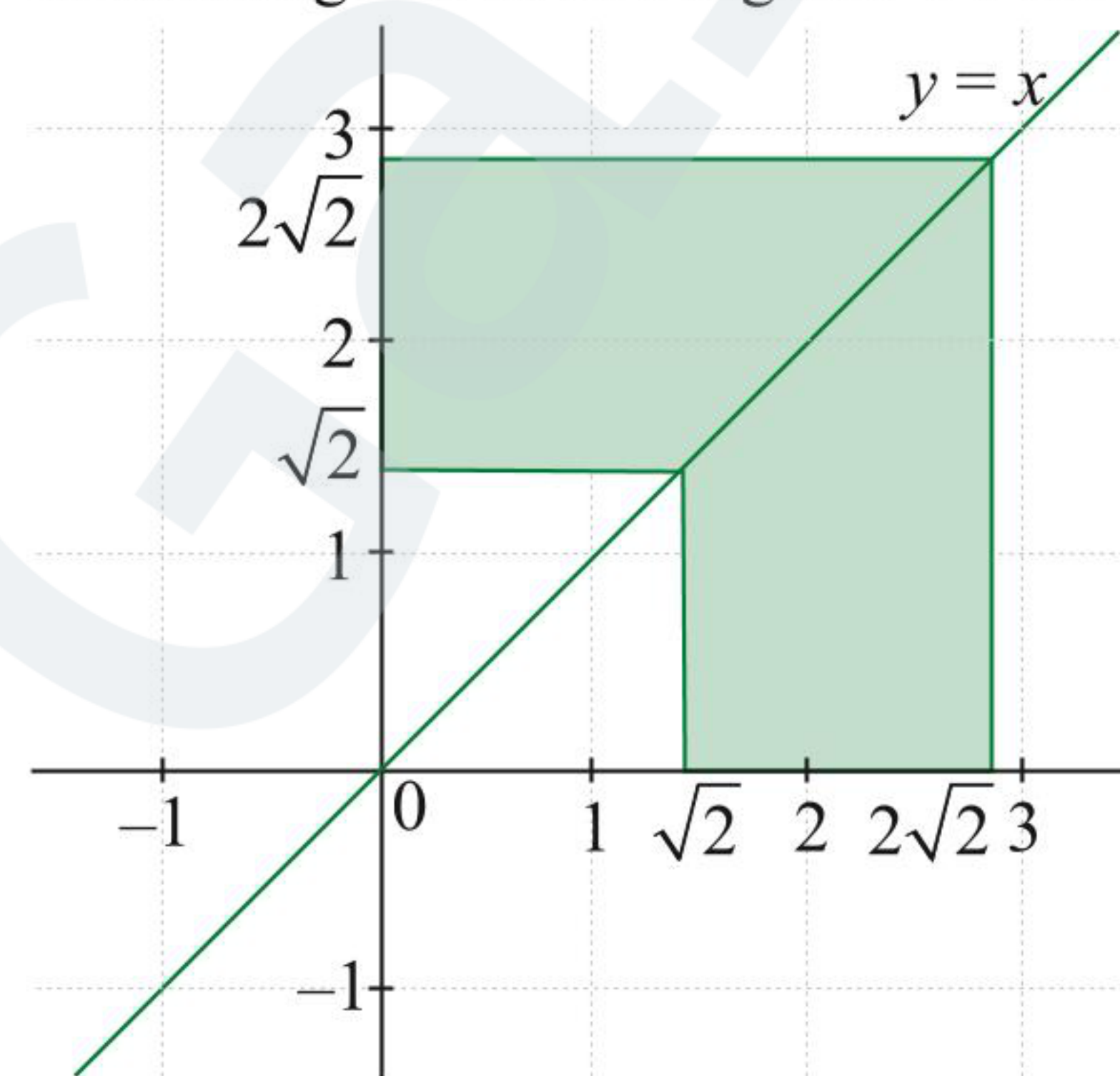
$$2\sqrt{2} \leq y - x + x + y \leq 4\sqrt{2}$$

$$\text{or } \sqrt{2} \leq y \leq 2\sqrt{2}$$

The region of points satisfying these inequalities is

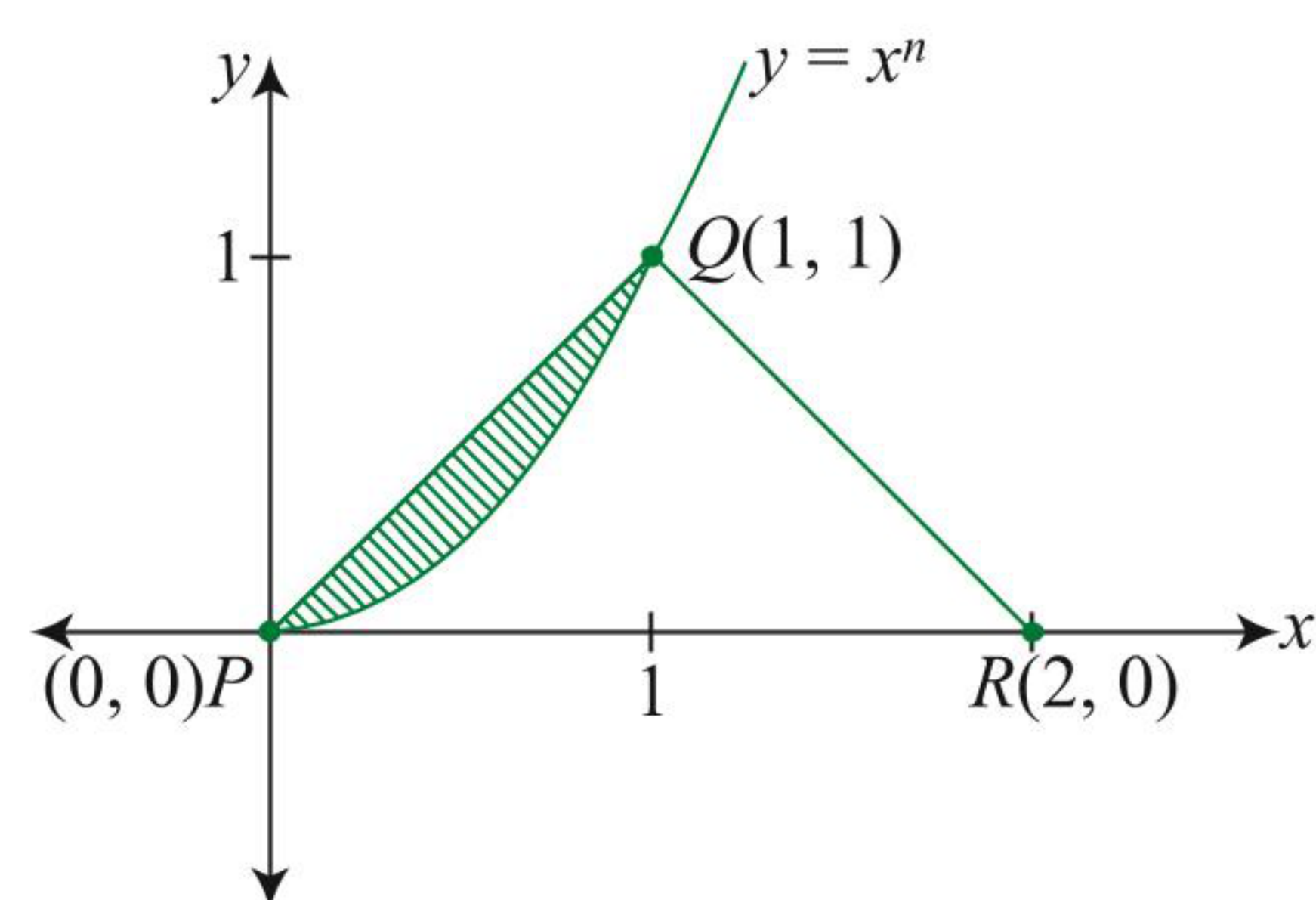


Combining above two regions we have



$$\begin{aligned} \text{Area of the shaded region} &= ((2\sqrt{2})^2 - (\sqrt{2})^2) \\ &= 8 - 2 = 6 \text{ sq. units} \end{aligned}$$

2. (4)



According to the question,

$$\int_0^1 (x - x^n) dx = \frac{3}{10} \left(\frac{1}{2} \times 2 \times 1 \right)$$

$$\Rightarrow \left| \frac{x^2}{2} - \frac{x^{n+1}}{n+1} \right|_0^1 = \frac{3}{10}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{n+1} = \frac{3}{10}$$

$$\Rightarrow \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}$$

$$\Rightarrow n = 4$$